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A Quadratic-move Transitional Ensemble Markov Chain Monte Carlo strategy for the Physics-guided Stochastic Reliability Analysis under Limited Data

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The paper presents a novel Quadratic-Move Transitional Ensemble Markov Chain Monte Carlo sampling strategy designed to improve mixing performance in high-dimensional sampling spaces. The core contribution lies in leveraging a quadratic-move proposal mechanism, which demonstrates superior exploration capabilities in confined and highly-skewed sample spaces compared to traditional stretch-move and random-walk strategies. To empirically assess its mixing performance, the proposed sampler is first implemented on a two-dimensional Rosenbrock function which provides a confined and highly-skewed sampling space. Following which, to demonstrate the practical relevance and performance of the proposed sampling strategy in a high-dimensional setting, a case study is conducted on a jet engine turbine blade whose reliability analysis is done on its thermal stress response. A physics-guided Bayesian reliability analysis is carried out using sparse data on the Young's modulus, Poisson's ratio, and the pressure loads from both the pressure and suction sides of the blade, whose respective aleatory variability is to be characterised. Comparative assessments against the stretch-move Transitional Ensemble Markov Chain Monte Carlo sampler in both case studies show that the quadratic-move variant significantly enhances the sample mixing performance. The results affirm the potential of the proposed strategy to improve inference robustness and reliability predictions in high-stakes engineering systems under limited data.

Keywords: Bayesian updating, Approximate Bayesian Computation, Transitional Ensemble Markov Chain Monte Carlo, Quadratic-move, Reliability analysis, Physics-enhanced Machine Learning.

1. Introduction

For safety-critical systems such as aeroplanes, oil-rigs, and nuclear reactors, its safe operation is of the essence which, in turn, protects human lives. This motivates the need for reliability analysis, which seeks to quantify how likely the system will perform its intended function over a given period of time under given conditions (Jin et al. (2016)).

However, such analysis faces numerous challenges such as: 1) the limited availability of data; 2) the performance variability of the system; and 3) the temporal relevance of the data relative to the current operating state of the system of interest. This introduces the element of uncertainty into the reliability analysis to which a standard approach to address this would be through Bayesian updating (Prakash and Pandey (2025)).

A key component of Bayesian updating is the Monte Carlo sampling procedure, and an open research challenge is to improve the convergence of the posterior samples (i.e., mixing) and the sample estimates Lye and Marino (2023). Hence, the objective of the paper is to contribute to the state-of-the-art of Bayesian reliability updating by proposing a Quadratic-Move Transitional Ensemble Markov Chain Monte Carlo sampling strategy, with two research objectives: 1) improve the sample mixing performance, especially for highly-skewed and anisotropic posteriors; and 2) improve the polymorphic uncertainty characterisation associated with the reliability input variables.

2. Bayesian Updating Framework

The Bayesian updating framework is mathematically defined as (Lye et al. (2022); Lye (2023)):

$$P(\boldsymbol{\theta}|\boldsymbol{D}, M) \propto P(\boldsymbol{\theta}|M) \cdot P(\boldsymbol{D}|\boldsymbol{\theta}, M) \qquad (1)$$

whereby $\boldsymbol{\theta}$ represents the vector of inferred parameter(s), $\boldsymbol{D}$ represents the vector of observed data, M represents the model that is being considered for updating, $P(\boldsymbol{\theta}|M)$ represents the prior, $P(\boldsymbol{D}|\boldsymbol{\theta}, M)$ represents the likelihood function, and $P(\boldsymbol{\theta}|\boldsymbol{D}, M)$ represents the posterior. Detailed reviews on each of the above terms are found in Lye and Marino (2023); Lye et al. (2020, 2023).

To sample from the un-normalised posterior $P(\boldsymbol{\theta}|\boldsymbol{D}, M)$, Markov Chain Monte Carlo (MCMC) strategies are developed such as: the Affine-invariant Ensemble sampler (AIES) (Goodman and Weare (2010)); 2) the Transitional Markov Chain Monte Carlo (TMCMC) sampler (Ching and Chen (2007)); and 3) the Stretch-move Transitional Ensemble Markov Chain Monte Carlo (TEMCMC-SM) sampler (Lye et al. (2022)).

The research work proposes a Quadratic-move Transitional Ensemble Markov Chain Monte Carlo (TEMCMC-QM), on which details are provided in Section 2.1.

2.1. *Proposed Sampling Strategy*

The TEMCMC-QM sampler implements the AIES as the MCMC move-kernel to improve the mixing performance of the sampler and overcome the computational challenge of sampling from a highly-skewed and anisotropic posterior. It generates posterior samples via the "transitional" distributions P^j defined as (Ching and Chen (2007)):

$$P^j \propto P(\boldsymbol{\theta}|M) \cdot P(\boldsymbol{D}|\boldsymbol{\theta}, M)^{\beta_j} \qquad (2)$$

where $j = 0, 1, \ldots, m$ is the iteration number, and β_j is the tempering parameter such that $\beta_0 = 0 < \beta_1 < \cdots < \beta_m = 1$.

An important aspect is the transition step size $\Delta\beta_j$ which affects how gradual the transition is from the prior to the final posterior. Following the work by Ching and Chen (2007), it was determined that the optimal value of $\Delta\beta_j$, and hence the value of β_j, is determined through solving the following optimisation problem:

$$\beta_{j+1} = \underset{\beta_{j+1}}{argmin} \left\{ \left| \frac{\sigma\left(P(\boldsymbol{D}|\boldsymbol{\theta}_i, M)^{\Delta\beta_j}\right)}{\mu\left(P(\boldsymbol{D}|\boldsymbol{\theta}_i, M)^{\Delta\beta_j}\right)} \right| - 1 \right\} \qquad (3)$$

where $\sigma(\bullet)$ is the standard deviation operator, $\mu(\bullet)$ is the mean operator, and $i = 1, \ldots, N$ is the sample index with N denoting the sample size.

The AIES sampler generates samples on an ensemble level $\vec{\boldsymbol{\theta}}_i^j$ comprising of N_c Markov chains: $\vec{\boldsymbol{\theta}}_i^j = \{\boldsymbol{\theta}_{1,i}^j, \boldsymbol{\theta}_{2,i}^j, \ldots, \boldsymbol{\theta}_{N_c-1,i}^j, \boldsymbol{\theta}_{N_c,i}^j\}$. In practice, it is required that $N_c \geq 2 \times N_d$ (Goodman and Weare (2010)). For a given iteration i of the AIES, the proposed sample $\boldsymbol{\theta}_{k,i}^{j*}$ for the k^{th} chain (for $k = 1, \ldots, N_c$) is generated via the Quadratic-move proposed by Militzer (2023):

$$\boldsymbol{\theta}_{k,i}^{j*} = w_k \cdot \boldsymbol{\theta}_{k,i}^j + w_m \cdot \boldsymbol{\theta}_{m,i}^j + w_n \cdot \boldsymbol{\theta}_{n,i}^j \qquad (4)$$

for which the statistical weights $w_k = L(t'_k; t_k, t_m, t_n)$, $w_m = L(t'_k; t_m, t_n, t_k)$, and $w_n = L(t'_k; t_n, t_k, t_m)$ where:

$$L(x; x_0, x_1, x_2) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} \qquad (5)$$

Following Militzer (2023), $t_m = -1$, $t_n = 1$, while both t'_k and t_k are assigned to follow a Normal distribution with mean 0 and standard deviation $a = 1.5$. This ensures that the statistical weights sum to one, and that the balance condition is satisfied: $P(\boldsymbol{\theta}_{k,i}^j \rightarrow \boldsymbol{\theta}_{k,i}^{j*}) = P(\boldsymbol{\theta}_{k,i}^{j*} \rightarrow \boldsymbol{\theta}_{k,i}^j)$. From there, $\boldsymbol{\theta}_{k,i}^{j*}$ is accepted with a probability α^j defined as (Militzer (2023)):

$$\alpha^j = \min\left[1, \frac{P(\boldsymbol{\theta}_{k,i}^{j*}|\boldsymbol{D}, M)}{P(\boldsymbol{\theta}_{k,i}^j|\boldsymbol{D}, M)} \cdot |w_i|^{N_{\text{dim}}}\right] \qquad (6)$$

where N_{dim} is the dimensionality of the epistemic sample space.

To moderate the acceptance rates of the TEMCMC-QM sampler, the scaling parameter u of the AIES needs to be controlled at each iteration j. At $j = 1$, an initial value of $u = 1.5$ is set (Militzer (2023)). From this initial value, the subsequent step-size u^{j+1} is computed after the

MCMC step:

$$u^{j+1} = u^j \cdot \exp\left[\alpha^j - \alpha_{\text{tr}}\right] \tag{7}$$

where α^j is the overall acceptance rate of the AIES sampler at iteration j, while α_{tr} is the target acceptance rate defined as (Roberts and Rosenthal (2001)):

$$\alpha_{\text{tr}} = \frac{0.21}{N_{\text{dim}}} + 0.23 \tag{8}$$

The implementation of the TEMCMC-QM sampler is summarised in Pseudoalgorithm 1, while the analyses in the paper are performed using a computing processor with specifications: 16.0 GB RAM, and an Intel(R) Core(TM) i5-12500 3.0 GHz processor.

3. Numerical Study: Rosenbrock Function

The study presents the Rosenbrock function:

$$f(x_1, x_2) = \frac{10000 \cdot \left(x_2 - {x_1}^2\right) + (1 - x_1)^2}{20} \tag{9}$$

whose contour plot is illustrated in Figure 1. The objective of the study is to compare the mixing performance of between the TEMCMC-SM and the proposed TEMCMC-QM sampler in sampling from a highly-skewed and constrained sample space.

3.1. *Problem set-up*

The inferred parameters are $\boldsymbol{\theta} = (x_1, x_2)$, with the prior assigned a Uniform distribution with the corresponding bounds: $x_1 \in [-4, 6]$, $x_2 \in [-2, 35]$. The likelihood function is defined as:

$$P(\boldsymbol{D}|\boldsymbol{\theta}, M) = \exp\left[-f(x_1, x_2)\right] \tag{10}$$

To obtain sufficient statistics over the total sampling iterations and the sampling times elapsed by the TEMCMC-SM and the TEMCMC-QM samplers, the sampling procedure is repeated $N_{\text{tr}} = 1000$ times.

3.2. *Results and Discussions*

A quantitative measure of mixing performance is $\Delta\beta_j$, which in turn affects the total sampling iterations. A larger $\Delta\beta_j$, therefore a smaller number of sampling iterations, indicates an improved sample mixing performance by the sampler.

The statistics over the total sampling iterations across N_{tr} runs for both the TEMCMC-SM and the TEMCMC-QM samplers is presented as a bar chart in Figure 1. The results indicate that the proposed TEMCMC-QM sampler has generally an improved sample mixing performance over the TEMCMC-SM sampler given that the former has significant higher number of runs (i.e., by nearly eight-fold) with 6 sampling iterations than the latter.

However, the total sampling time for each run is generally higher in the case of the proposed TEMCMC-QM sampler than the TEMCMC-SM sampler as indicated by the histograms illustrated in Figure 1. The sampling time statistics for both the TEMCMC-SM and the proposed TEMCMC-QM samplers is presented in Table 7. The slight increase in sampling time by the TEMCMC-QM sampler is due to the sampler requiring a sample draw for both variables t'_k and t_k, whereas the sample draw procedure is only required for the stretch factor variable in the case of the TEMCMC-SM sampler.

Table 1. Statistics on the sampling time elapsed for each sampler.

Sampler	TEMCMC-SM	TEMCMC-QM
Mean $[s]$	1.18	1.61
Standard deviation $[s]$	0.25	0.35

Finally, to verify the resulting posterior sample distribution obtained by the proposed TEMCMC-QM sampler, the Hellinger distance, recently implemented in Lye et al. (2025), is used to quantify the similarity with that obtained by the TEMCMC-SM sampler, defined as (Hellinger (1909)):

$$d_H(\boldsymbol{D}, \boldsymbol{D}^{\text{sim}}) = \sqrt{\frac{\sum_{x_{n_d}=1}^{N_{\text{bin}}} \cdots \sum_{x_1=1}^{N_{\text{bin}}} \left(\sqrt{p_{\boldsymbol{D}}(\boldsymbol{x})} - \sqrt{p_{\boldsymbol{D}^{\text{sim}}}(\boldsymbol{x})}\right)^2}{2}} \tag{11}$$

Algorithm 1 Proposed TEMCMC-QM sampler algorithm

procedure (Generate N samples from $P(\boldsymbol{\theta}|\boldsymbol{D}, M)$)
  Set $j = 0$ and $\beta_j = 0$ ▷ Initialise
  Draw N initial sample set: $\boldsymbol{\theta}_i^j \sim P(\boldsymbol{\theta}|M)$
  Set $u^{j+1} = 1.5$ ▷ Set initial value of scaling parameter
  while $\beta_j < 1$ **do** ▷ Main sampling loop
    Set $j = j + 1$
    Compute $\Delta\beta_j$ via Eq. (3), and P^j via Eq. (2) ▷ Set the "transitional" distribution
    Resample N samples: $\boldsymbol{\theta}_i^j \sim \hat{w}_i^j = \frac{P(\boldsymbol{D}|\boldsymbol{\theta}_i^j, M)^{\Delta\beta_j}}{\sum_{i=1}^{N} P(\boldsymbol{D}|\boldsymbol{\theta}_i^j, M)^{\Delta\beta_j}}$
    Set $\boldsymbol{\theta}_i^j = \boldsymbol{\theta}_{i,1}^j$ in ensemble $\vec{\boldsymbol{\theta}}_1^j$ ▷ Initiate ensemble
    Update $\vec{\boldsymbol{\theta}}_1^j$ with 1 iteration of AIES with Quadratic-move via Eq. (5) and (6) ▷ MCMC step
    Set samples in ensemble $\vec{\boldsymbol{\theta}}_2^j$ as samples of P^j ▷ Update samples in the ensemble
    Compute u^{j+1} via Eq. (7) ▷ Tuning the scaling parameter
  end while
  Compute $P(\boldsymbol{D}|M) \approx \prod_{j=1}^{m} \frac{1}{N} \cdot \sum_{i=1}^{N} P(\boldsymbol{D}|\boldsymbol{\theta}_i^j, M)^{\Delta\beta_j}$ ▷ Compute the model evidence term
end procedure

where $\boldsymbol{x} = (x_1, \ldots, x_{n_d})$ denotes the input bin vector, n_d denotes the total number of data components, and N_{bin} is the total number of bins used to approximate the probability density functions of $\boldsymbol{D}$ and $\boldsymbol{D}^{\text{sim}}$ which are denoted respectively as $p_{\boldsymbol{D}}(\bullet)$ and $p_{\boldsymbol{D}^{\text{sim}}}(\bullet)$ respectively. To compute N_{bin}, the adaptive-binning approach proposed by Zhao et al. (2022) is implemented:

(1) Compute $\Delta^{\text{sim}} = \max\left(\max|\boldsymbol{D}_{i,m}^{\text{sim}} - \boldsymbol{D}_{j,m}^{\text{sim}}|\right)$; where $i, j = 1, \ldots, N$, $m = 1, \ldots, n_d$, and N denotes the total samples obtained from the posterior.
(2) Compute the Euclidean distance between the data set $\boldsymbol{D}$ and $\boldsymbol{D}^{\text{sim}}$: $d_E(\boldsymbol{D}, \boldsymbol{D}^{\text{sim}}) = \sqrt{(\overline{\boldsymbol{D}^{\text{sim}}} - \overline{\boldsymbol{D}}) \cdot (\overline{\boldsymbol{D}^{\text{sim}}} - \overline{\boldsymbol{D}})^{\text{T}}}$; where $\overline{\boldsymbol{D}^{\text{sim}}}$ and $\overline{\boldsymbol{D}}$ are the respective means of $\boldsymbol{D}^{\text{sim}}$ and $\boldsymbol{D}$.
(3) Compute $w = \frac{\log\left(\Delta^{\text{sim}}+1\right)}{\max\left(N^{1/3}, {N_{\text{obs}}}^{1/3}\right)} \times \exp\left(d_E\right)$
(4) Compute $N_{\text{bin}} = \lceil \frac{\Delta^{\text{sim}}}{w} \rceil$, and ensure that $N_{\text{bin}} \in \left[2, \lceil \frac{\max(N, N_{\text{obs}})}{10} \rceil\right]$.

The resulting Hellinger distance between the posterior sample distribution obtained by both samplers is 3.36×10^{-3}, which indicates a strong degree of similarity and verifies the results obtained by the proposed TEMCMC-QM sampler.

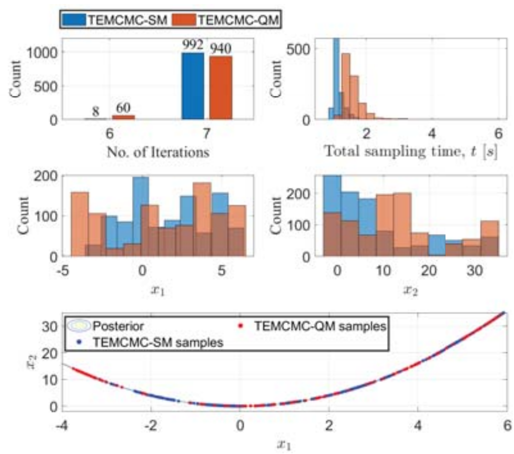

Fig. 1. Results on the mixing performance statistics, sampling time, and sampling performance for the TEMCMC-SM and the TEMCMC-QM samplers.

4. Case Study: Jet Engine Turbine Blade

The study presents a jet engine turbine blade component with material properties: Young's modulus E; Poisson's ratio ν; and coefficient of thermal expansion κ. The physics-based finite element

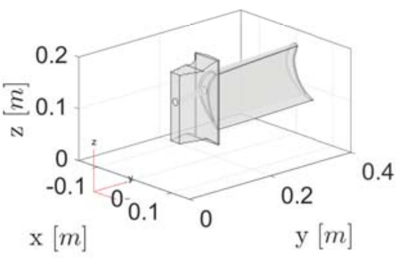

Fig. 2. The jet engine turbine blade.

model of the blade is available in the Partial Differential Equation Toolbox of Matlab R2022b. The boundary condition is set such that the surface of the root in contact with the other metal is fixed.

The component is subjected to pressure loads on its pressure and suction sides, denoted as p_1 and p_2 respectively. From there, the component's performance reliability is to be assessed, under limited data, given the following performance function g:

$$g = \sigma_{\text{tr}} - \sigma_{\text{max}}(E, \nu, \kappa, p_1, p_2) \qquad (12)$$

where $\sigma_{\text{max}}(\bullet)$ is the maximum von Mises stress of the blade, while $\sigma_{\text{tr}} = 0.34\ GPa$ is the defined regulatory threshold value set for this problem.

The goals of the study are: 1) to characterise the polymorphic uncertainty of the input variables p_1 and p_2 based on the limited data available; 2) to estimate the component's failure probability $P(g < 0)$ along with its uncertainty; and 3) to evaluate the sampling performance of the proposed TEMCMC-QM sampler under high dimensions (i.e., 17 dimensions).

4.1. *Problem set-up*

Details on the input parameters and variables are presented in Table 2. To simulate the case of limited data for the stochastic reliability analysis, 20 samples of the following quantities are obtained: E, ν, p_1, and p_2.

The likelihood function set-up is defined as (Bi et al. (2019)):

Table 2. Details on the input parameters/variables.

Input	Distribution / Value	Unit
E	Normal($\mu = 220 \times 10^9, \sigma = 220 \times 10^8$)	Pa
ν	Normal($\mu = 0.30, \sigma = 0.015$)	–
κ	1.25×10^{-7}	$1/K$
p_1, p_2	Gumbel($\mu = 5 \times 10^5, \sigma = 7.5 \times 10^4$)	Pa
ρ	0.2 (Correlation coefficient between p_1 and p_2)	–

$$P(\boldsymbol{D}|\boldsymbol{\theta}, M) = \exp\left[-\left(\frac{d_H(\boldsymbol{D}, \boldsymbol{D}^{\text{sim}})}{\varepsilon}\right)^2\right] \qquad (13)$$

where $d_H(\bullet)$ is the Hellinger distance function quantifying the statistical difference between the distribution of the data $\boldsymbol{D}$ and the output model samples $\boldsymbol{D}^{\text{sim}} = M(\boldsymbol{\theta})$, while ε is the width-factor which controls the centralization degree of the resulting posterior.

To simulate the model-form uncertainty over the distribution of E, ν, p_1, and p_2, the Staircase Density Function (SDF) is used to characterise their respective variability (Crespo et al. (2018)). The SDF is a meta-model which can model a distribution from a data-set based on the statistical moments m_r, for $r = 1, \ldots, 4$:

$$m_r = \int_{\underline{z}}^{\overline{z}} (z - \mu)^r \cdot f_z(z) \cdot dz \qquad (14)$$

where the integration limits $\Delta_z = [\underline{z}, \overline{z}]$ is the bounded support set over the staircase random variable z, the function f_z is the probability density function, and μ is the expected value of the data variable z. The set of parameters that make up the a staircase random variable are: $\boldsymbol{\theta}_z = \{\Delta_z, \mu, m_2, m_3, m_4\}$, which are bounded by a set of feasibility constraints such that (Crespo et al. (2018)):

$$\mu \in [\underline{z}, \overline{z}]; \quad m_2 \in \left[0, \frac{(\overline{z} - \underline{z})^2}{4}\right];$$
$$m_3 \in \left[-\frac{(\overline{z} - \underline{z})^3}{6\sqrt{3}}, \frac{(\overline{z} - \underline{z})^3}{6\sqrt{3}}\right]; \quad m_4 \in \left[0, \frac{(\overline{z} - \underline{z})^4}{12}\right] \qquad (15)$$

The bounded support Δ_z of the SDF for each variable is determined and presented in Table 3.

Table 3. Details on Δ_z and ε for each input parameter/variable.

Variable	E	ν	p_1, p_2
Δ_z	$[1.0, 3.0] \cdot 10^{11}\ Pa$	$[2.0, 4.0] \cdot 10^{-1}$	$[2.0, 6.5] \cdot 10^{5}\ Pa$
ε	0.07	0.08	0.12

The choice on Δ_z is such that it provides sufficiently large enough bounds on the SDF shape parameters μ, m_2, m_3, and m_4, and hence, sufficient degrees of freedom for the SDF to model the distribution of the respective aleatory variable. For the respective variable, the prior is set as a Uniform distribution for the inferred parameters $\boldsymbol{\theta} = \{\mu, m_2, m_3, m_4\}$, with the bounds defined by Table 3 and Eq. (15). For the likelihood function, the width parameter ε associated with each variable is defined in Table 3, and is chosen such that it provides sufficient convergence over the posterior estimates.

4.2. *Results and Discussions*

The resulting posterior sample distributions of the inferred parameters, obtained by the respective sampler, are presented in Figure 3. From which, the 20 % two-sided credible interval for each inferred parameter is obtained, and the numerical results are presented in Table 4.

The credible intervals form a 17-dimensional hyper-rectangle of the epistemic space. Based on the resulting credible intervals on the inferred shape parameters of the SDF for the respective input variable, a Double-loop Monte Carlo is implemented to construct the p-boxes with $\{N_a, N_e\} = \{10000, 1000\}$, where N_a denotes the number of aleatory realizations from the SDF, while N_e is the number of epistemic realizations from the epistemic space. The computational time for each p-box lies in the interval of $[3, 5]\ s$, and the resulting p-boxes for the respective variable are presented in Figure 4. As seen from the figure, the resulting p-boxes obtained with the proposed TEMCMC-QM sampler generally agrees with that obtained by the TEMCMC-SM sampler, and they both generally enclose the ECDF of the data, which provides a form of verification.

From there, the physics-guided stochastic reliability analysis is performed via a Double-loop Monte Carlo procedure over the physics-based finite element model of the blade with $\{N_a, N_e\} = \{500, 500\}$ to which the results are presented in Table 5 and verified against the true result obtained via Monte Carlo simulation.

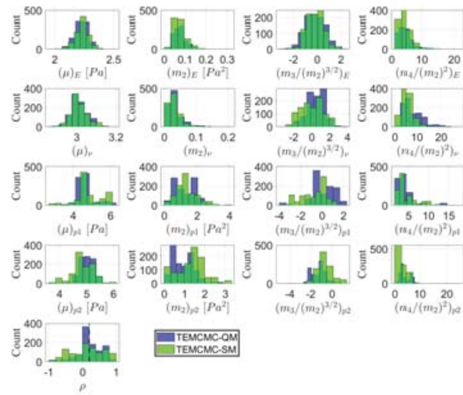

Fig. 3. Results on the posterior distribution obtained via the respective samplers.

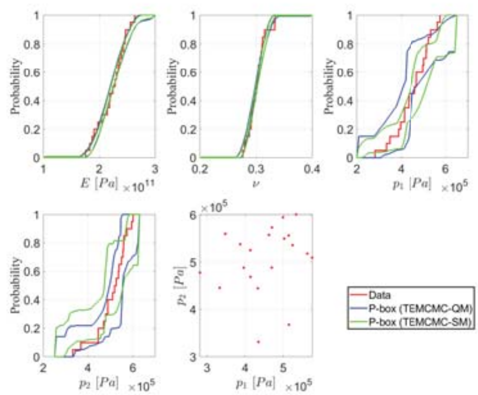

Fig. 4. Results on the data ECDF, and the p-boxes obtained via the respective samplers.

4.3. *Further Discussions*

Further analysis is done to assess the mixing performance of the proposed TEMCMC-QM sam-

Table 4. Results on the 20 % two-sided credible intervals over the inferred parameters.

Variable	Sampler	μ	m_2	$m_3/(m_2)^{3/2}$	$m_4/(m_2)^2$	ρ
E	TEMCMC-QM	$[2.21, 2.24] \cdot 10^{11}$	$[6.46, 7.92] \cdot 10^{20}$	$[-0.47, 0.04]$	$[3.59, 4.61]$	–
	TEMCMC-SM	$[2.22, 2.25] \cdot 10^{11}$	$[6.22, 7.66] \cdot 10^{20}$	$[-0.40, 0.07]$	$[3.46, 4.40]$	–
ν	TEMCMC-QM	$[3.00, 3.01] \cdot 10^{-1}$	$[2.41, 3.31] \cdot 10^{-4}$	$[0.21, 0.89]$	$[5.44, 7.47]$	–
	TEMCMC-SM	$[3.00, 3.02] \cdot 10^{-1}$	$[2.19, 2.94] \cdot 10^{-4}$	$[-0.36, 0.53]$	$[4.89, 6.22]$	–
p_1	TEMCMC-QM	$[4.60, 4.64] \cdot 10^{5}$	$[0.89, 1.61] \cdot 10^{10}$	$[-0.42, 0.45]$	$[2.83, 3.74]$	$[0.15, 0.24]$
	TEMCMC-SM	$[4.45, 4.77] \cdot 10^{5}$	$[1.22, 1.36] \cdot 10^{10}$	$[-0.90, -0.06]$	$[2.38, 3.30]$	$[0.10, 0.25]$
p_2	TEMCMC-QM	$[4.94, 5.10] \cdot 10^{5}$	$[0.64, 1.12] \cdot 10^{10}$	$[-1.43, -0.97]$	$[2.87, 4.27]$	–
	TEMCMC-SM	$[4.68, 4.94] \cdot 10^{5}$	$[1.28, 1.52] \cdot 10^{10}$	$[-0.79, -0.58]$	$[1.67, 2.96]$	–

Table 5. Results on the physics-guided stochastic reliability analysis.

	Monte Carlo	TEMCMC-SM	TEMCMC-QM
Probability	0.003	$[0, 0.008]$	$[0, 0.010]$

pler. To obtain sufficient statistics over the total sampling iterations and the sampling times elapsed by the TEMCMC-SM and the TEMCMC-QM samplers, the posterior sampling procedure is repeated $N_{\text{batch}} = 100$ times. The resulting statistics over the total sampling iterations is presented as a bar chart in Figure 5 and numerically in Table 7, where it is seen that the proposed TEMCMC-QM sampler mostly take between 15 and 16 iterations whereas the TEMCMC-SM sampler mostly take between 16 and 17 iterations. This indicates the improved mixing performance of the TEMCMC-QM sampler over the TEMCMC-SM sampler. In addition, the statistics of the sampling time is presented as a histogram in Figure 5 and numerically in Table 7, where it is seen that the sampling time is generally lesser for the TEMCMC-QM sampler than the TEMCMC-SM sampler. This is attributed to the fact that for most of the runs, the TEMCMC-QM sampler samples from the posterior in less iterations than the TEMCMC-SM sampler.

Table 6. Statistics on the total sampling iteration for each sampler.

Sampler	TEMCMC-SM	TEMCMC-QM
Mean	16.31	15.61
Standard deviation	1.29	1.25

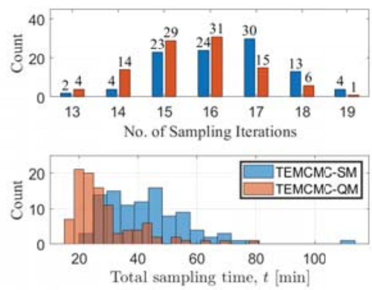

Fig. 5. Statistics on the total sampling iteration and sampling time by the respective samplers across $N_{\text{batch}} = 100$ runs.

Table 7. Statistics on the sampling time elapsed for each sampler.

Sampler	TEMCMC-SM	TEMCMC-QM
Mean [min]	42.95	28.91
Standard deviation [min]	14.52	11.69

5. Conclusion

The paper has proposed the TEMCMC-QM sampler with improved mixing performance in high-dimensional stochastic reliability analyses under data scarcity. This is achieved via a Quadratic-move strategy to improve sample exploration in highly-skewed and constrained spaces, as well as an adaptive-tuning algorithm to moderate the acceptance rates within optimal bounds. The performance of the TEMCMC-QM sampler is then assessed against the TEMCMC-SM sampler in two case studies: 1) the Rosenbrock function; and 2) the reliability analysis of a jet engine turbine blade. Results in both case studies verified the results obtained by the TEMCMC-QM sampler against that of the TEMCMC-SM sampler, and demonstrated the significant improvement in the mixing performance of the TEMCMC-QM sampler.

Future works may consider the following to further improve the mixing performance: 1) proposing a new selection criteria on the transition step of the sampler; 2) deriving the optimal sample acceptance rate; and 3) improving on the Markov Chain Monte Carlo move strategy to provide better exploration of the sample space.

The MATLAB codes to the analysis presented in the paper are accessible via: `https://github.com/Adolphus8/Transitional_Ensemble_MCMC`

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