Proc. of the European Safety and Reliability Conference (ESREL 2026)
Edited by José C. Matos, Paulo B. Lourenço, Michael Beer, Daniel Oliveira,
Edoardo Patelli, Hélder S. Sousa, Rui A. Silva and Monica Santamaria

doi: 10.3850/ESREL2026061419_esrel26-p26249-cd

A Recursive Algorithm for Computing Joint Probabilities in a Bayesian Network, Illustrated by its Application to Safety Analysis of an Industrial Facility

Jacek Malinowski

Systems Research Institute, Polish Academy of Sciences, Poland. E-mail: jacek.malinowski@ibspan.waw.pl

Bayesian networks are direct acyclic graphs used to model and analyze dependencies between random events linked by cause-effect relationships. Among other applications, they provide a suitable tool for both qualitative and quantitative analysis of safety-related scenarios in industrial environments. The qualitative analysis consists in a detailed examination of the network structure. A top-down examination aims to predict potential consequences of hazard-triggering initial events. In turn, a bottom-up one allows for identifying possible root causes (initial events) of hazards, damages, accidents, etc. (secondary events). The purpose of the quantitative approach is to assess the chances of various scenarios. It inherently involves calculating conditional joint probabilities of several secondary events (effects) given the occurrence of particular initial events (causes). Apart from these "forward" probabilities, we may also need to compute the "backward" ones, which are the joint probabilities of possible causes given that particular effects occur. However, the computing time is likely to increase rapidly with the network size and complexity. This paper presents a novel method for computing the above probabilities in a reasonable time for networks of a small to medium width, employing a recursive formula. Here, the network width is defined as the maximum number of nodes in one layer, where the definition of a layer is given further in the paper. The presented method is illustrated with the safety model of a biogas production unit in a biogas plant.

Keywords: Bayes network, cause-effect relation, forward/reverse probability, extended CPT, recursive procedure.

1. Introduction

The main result presented here is a recursive method for computing the forward and reverse probabilities in a Bayes network (BN). The former (FP) is defined as the probability that certain effects are potentially caused by the given triggering events, while the latter (RP) is defined as the probability that certain triggering events are potential causes of the given effects. Various events occurring in the environment modeled by a BN are represented by random variables assigned to the network nodes. Thus, FP is the probability that certain non-initial variables take particular values, given the values of selected initial variables. In turn, RP is the above probability, where „initial" and "non-initial" are swapped with each other. Both types of probabilities can be useful in analyzing complex cause-effect relations in systems modeled by BNs. The aforementioned method computes FPs, assuming that the nodes of a BN are divided into layers such that the nodes in the lower layer have parents in the higher layer or are located between nodes with parents in the higher layer. The partitioning of nodes into layers depends on the network structure and selection of initial and non-initial nodes for the cause-effect analysis. Meanwhile, RPs are obtained from FPs using the Bayes formula, i.e. $\Pr(A|B)=\Pr(B|A)\Pr(A)/\Pr(B)$, where A and B respectively denote the causal and effectual events. Applying recursion reduces the significant computational complexity occurring when FPs are computed directly from the formula for the factorization of the probability of joint events. It also allows us to determine certain "intermediate" probabilities that can prove useful for the BN analysis.

This work falls into the category of causal analysis in the safety and reliability context using Bayes networks. Clearly, there is abundant literature on this topic, including Gansch et al. 2025, Lu et al. 2023, Tchórzewska-Cieślak 2017, Vázquez-Aguirre et al. 2022, Wang et al. 2025, to mention some recent and older relevant papers. For an introduction to Bayes networks, as well as methods for their analysis and learning the structure and/or parameters, the reader is referred to Ding 2010, Heckerman 2022, Jensen and Nielsen 2007, Kjaerluff and Madsen 2013, Scutari and Denis 2021. Nevertheless, despite extensive literature, it is rather difficult to find efficient algorithms for computing the forward and reverse probabilities mentioned above, needed for the quantitative analysis of BN-modeled systems. The author believes that his present paper will, in a small way, contribute to the methodology of modeling and analyzing causal relationships expressed by Bayes networks.

2. Recursive computation of the forward probabilities

To formally define FPs and RPs, the following notation is adopted:

n – number of nodes (variables) in a BN
X_j – node (variable) in BN, $j=1,\ldots,n$
$V(j)$ – set of values of X_j
v_j – one of the values of X_j
$<$ – precedence relation between nodes, i.e. $X<Y$ (X precedes Y) if there is a path from X to Y
A – selected initial nodes representing causes
B – selected non-initial nodes representing effects; we assume that for each $X\in A$ there exists $Y\in B$ such that $X<Y$, and for each $Y\in B$ there exists $X\in A$ such that $X<Y$
C – nodes belonging to B or preceding its nodes, i.e. $X\in C$ if $X\in B$ or $X<Y$ for certain $Y\in B$; note that $A\subset C$ and some initial nodes in C may be outside A
$I(A)$, $I(B)$, $I(C)$ – sets of indexes of nodes in A, B and C respectively
$P(j)$ – set of indexes of X_j's parents
$\Pr[X_j=v_j, j\in I(B) \mid X_j=v_j, j\in I(A)]$ – forward prob.
$\Pr[X_j=v_j, j\in I(A) \mid X_j=v_j, j\in I(B)]$ – reverse prob.

FP can be straightforwardly computed using the following factorization formula:

$$\Pr(E_1 \cap \ldots \cap E_m | E_{m+1} \cap \ldots \cap E_n) = = \prod_{i=1}^{m} \Pr(E_i \mid E_{i+1} \cap \ldots \cap E_n), \tag{1}$$

where $E_1,\ldots,E_n$ are arbitrary events and $m\leq n$. If $m=n$ then the left-hand side of Eq. (1) is equal to $\Pr(E_1\cap\ldots\cap E_m)$ and the last factor in the product on the right-hand side is equal to $\Pr(E_n)$. For the (straightforward) proof of Eq. (1), as well as Eqs. (2)-(5), see Jensen and Nielsen 2007 or Kjaerluff and Madsen 2013. Applying Eq. (1), we obtain:

$$\Pr[X_j = v_j, j \in I(B) \mid X_j = v_j, j \in I(A)] =$$
$$= \sum_{\substack{v_j\in V(j),\\ j\in I(C)\setminus\\ I(A\cup B)}} \Pr[X_j = v_j, j \in I(C\setminus A) \mid X_j = v_j, j \in I(A)]$$
$$= \sum_{\substack{v_j\in V(j),\\ j\in I(C)\setminus\\ I(A\cup B)}} \prod_{i\in I(C\setminus A)} \Pr[X_i = v_i \mid X_h = v_h, h \in P(i)] \tag{2}$$

If X_i is an initial variable, then it has no parents and $\Pr[X_i=v_i \mid X_h=v_h, h\in P(i)]=\Pr(X_i=v_i)$. The first equality in Eq. (2) follows from the fact that the events $\{X_j=v_j\}$, $v_j\in V(j), j\in I(C)$ constitute an exhaustive set of events. The second is a consequence of Eq. (1) and the rule stating that a variable in a BN is independent of its non-descendants, given the values of its parents. It should be noted that the probabilities in the final expression in Eq. (2) are stored in the CPTs of the variables X_i, $i\in I(C)$, which are assumed to be known.

The reverse probability $\Pr[X_j=v_j, j\in I(A) \mid X_j=v_j, j\in I(B)]$ can be computed using the forward probability and the Bayes formula, i.e. we have:

$$\Pr[X_j = v_j, j \in I(A) \mid X_j = v_j, j \in I(B)] =$$
$$= \Pr[X_j = v_j, j \in I(B) \mid X_j = v_j, j \in I(A)] \times$$
$$\times \frac{\Pr[X_j=v_j,\, j\in I(A)]}{\Pr[X_j=v_j,\, j\in I(B)]} \tag{3}$$

The initial variables in a BN are independent, hence

$$\Pr[X_j = v_j, j \in I(A)] = = \prod_{j\in I(A)} \Pr(X_j = v_j), \tag{4}$$

while $\Pr[X_j, j\in I(B)]$ is obtained using the law of total probability:

$$\Pr[X_j = v_j, j \in I(B)] = \sum_{\substack{v_i\in V(i),\\ i\in I(A)}} \Pr[X_j = v_j, j \in I(B) \mid X_i = v_i, i \in I(A)] \times \Pr[X_i = v_i,\ i \in I(A)] \tag{5}$$

Note that to compute the reverse probability $\Pr[X_j, j\in I(A) \mid X_j, j\in I(B)]$ we need the forward probabilities $\Pr[X_j, j\in I(B) \mid X_j, j\in I(A)]$ for all values of $X_j, j\in I(A)$.

As can be seen, computing the forward probabilities $\Pr[X_j=v_j, j\in I(B) \mid X_j=v_j, j\in I(A)]$ from Eq. (2) involves summation over all values of the variables preceding those in B, except those in A. However, FPs can be computed using a recursive procedure developed by the author, which significantly reduces the number of summations performed in a single step and finds certain intermediate forward probabilities, also useful in BN analysis. As a prerequisite for this procedure, the nodes of C (see the definition at the beginning of this section) are divided into disjoint ordered layers such that initial nodes make up layer 0 and the nodes in layer q have parents only in layers q–1 or q (their own layer), where $q\geq 1$. The detailed definitions of these layers are given below.

$U(B)$ – set composed of nodes belonging to B or preceding its nodes (another notation for C)
$U_0(B)$ – layer 0 of $U(B)$, i.e. subset of $U(B)$ composed of initial nodes
$U_1(B)$ – layer 1 of $U(B)$, i.e. subset of $U(B)$ composed of nodes with parents in $U_0(B)$, or nodes located between such nodes; the nodes in $U_1(B)$ with parents in $U_0(B)$ are the upper border nodes in layer 1
$W_1(B)$ – nodes in $U_1(B)$ with at least one child node in $U(B)\backslash U_1(B)$; they are the lower border nodes in layer 1, while their children in $U(B)\backslash U_1(B)$ are the upper border nodes in layer 2
$U_q(B)$ – layer q of $U(B)$, i.e. subset of $U(B)\backslash[U_1(B),\ldots,U_{q-1}(B)]$ composed of nodes with parents in $W_{q-1}(B)$, or nodes located between such nodes, where $q\geq 2$;
$W_q(B)$ – nodes in $U_q(G)$ with at least one child node in $U(B)\backslash[U_1(B),\ldots,U_q(B)]$, where $q\geq 2$; they are the lower border nodes in layer q, while their children in $U(B)\backslash[U_1(B),\ldots,U_q(B)]$ are the upper border nodes in layer q+1
$I_q(B)$, $J_q(B)$ – indexes of nodes in $U_q(B)$, $W_q(B)$ respectively, where $q\geq 0$

Let us note that $U_{k-1}(B)\neq\varnothing$ and $U_k(B)=\varnothing$ for certain $k(B)\geq 2$, in which case $W_{k-1}(B)=\varnothing$ and the nodes of C are divided into $k(B)$ disjoint layers $U_0(B),\ldots, U_{k-1}(B)$. Clearly, $I_0(B),\ldots, I_{k-1}(B)$ are the respective disjoint groups of indices.

The following theorem provides recursive formulas for computing forward probabilities. Once they are determined, the reverse probabilities can be computed from Eq. (3).

Theorem 1
Let B and A be groups of nodes (variables) such that B is composed of selected non-initial nodes and A is composed of selected initial nodes preceding those of B (each node in A precedes at least one node in B). Let $U_0(B),\ldots,U_{k-1}(B)$ be the subsets of $U(B)$ consisting of the nodes located respectively in layers $0,\ldots,k(B)-1$.

If $k(B)\geq 3$, i.e. $U(B)$ is divided into three or more layers, then the conditional probabilities $\Pr[X_j = v_j, j\in I(B) \mid X_j = v_j, j\in I(A)]$, where $v_j\in V(j)$, $j\in I(B)\cup I(A)$, can be computed from the following three recursive formulas:

$$R_1[v_j,\ j\in J_1(B)] =$$

$$= \Pr[X_j = v_j,\ j\in J_1(B) \mid X_j = v_j,\ j\in I(A)]$$

$$= \sum_{\substack{v_j\in V(j),\\ j\in[I_0(B)\cup I_1(B)]\backslash\\ [I(A\cup B)\cup J_1(B)]}} \downarrow$$

$$\prod_{i\in I_1(B)} \Pr[X_i = v_i \mid X_h = v_h,\ h\in P(i)] \times \times \prod_{i\in I_0(B)\backslash I(A)} \Pr(X_i = v_i) \tag{6}$$

which is the formula for step 1, next

$$R_{q+1}[v_j, j\in J_{q+1}(B)] =$$

$$= \Pr[X_j = v_j, j\in J_{q+1}(B) \mid X_j = v_j, j\in I(A)]$$

$$= \sum_{\substack{v_j\in V(j),\\ j\in[J_q(B)\cup I_{q+1}(B)]\backslash\\ [I(B)\cup J_{q+1}(B)]}} \downarrow$$

$$\prod_{i\in I_{q+1}(B)} \Pr[X_i = v_i \mid X_h = v_h, h\in P(i)] \times \times R_q[v_i,\ i\in J_q(B)] \tag{7}$$

which is the formula for steps $2,\ldots,k(B)-2$, i.e. for $q=1,\ldots,k(B)-3$. Note that Eq. (7) is valid for $k(B)\geq 4$, i.e. if $U(B)$ is divided into four or more layers. Below is the formula for the last step, i.e. for q equal to $k(B)-2$. For shortness, $k(B)$ is abbreviated to k.

$$R_{k-1}\left[v_j, j \in I_{k-1}(B) \cap I(B)\right] =$$

$$= \Pr\left(\begin{matrix} X_j = v_j, \\ j \in I_{k-1}(B) \cap I(B) \end{matrix} \middle| \begin{matrix} X_j = v_j, \\ j \in I(A) \end{matrix} \right)$$

$$= \sum_{\substack{v_j \in V(j), \\ j \in [J_{k-2}(B) \cup I_{k-1}(B)] \setminus \\ I(B)}} \downarrow$$

$$\prod_{i \in I_{k-1}(B)} \Pr[X_i = v_i \mid X_h = v_h, h \in P(i)] \times \times R_{k-2}[v_i, i \in J_{k-2}(B)] \tag{8}$$

The value R_{k-1} computed with Eq. (8) is the sought FP, i.e. it holds that

$$\Pr\left[X_j = v_j,\ j \in I(B) \mid X_j = v_j,\ j \in I(A)\right] =$$

$$= R_{k-1}\left[v_j,\ j \in I_{k-1}(B) \cap I(B)\right] \tag{9}$$

where k stands for $k(G)$, as in Eq. (8).

In the case $k(B)=2$, i.e. $U(B)=U_0(B) \cup U_1(B)$, FP is computed in one step as follows:

$$\Pr\left[X_j = v_j,\ j \in I(B) \mid X_j = v_j,\ j \in I(A)\right] =$$

$$= \sum_{\substack{v_j \in V(j), \\ j \in [I_0(B) \cup I_1(B)] \setminus \\ [I(A \cup B)]}} \downarrow$$

$$\prod_{i \in I_1(B)} \Pr[X_i = v_i \mid X_h = v_h,\ h \in P(i)] \times \times \prod_{i \in I_0(B) \setminus I(A)} \Pr(X_i = v_i) \tag{10}$$

The proof of Theorem 1, along with formulas (6)-(10), will be presented in the extended version of this paper, which is being prepared for publication.

Remark 1: When calculating the successive R_q, $q=1,\ldots,k(B)-1$, we assume that the values v_j of the variables X_j, $j \in I(A \cup B)$ are fixed. For this reason, the summations in Eqs. (6)-(8) and (10) run over the indexes that do not belong to $I(A \cup B)$.

Remark 2: Each conditional probability $R_q[v_j, j \in J_q(B)]$, $q=1,\ldots,k(B)-2$, and R_{k-1} $[v_j, j \in I_{k(B)-1} \cap I(B)]$ has to be computed for all relevant values $v_j \in V(j)$ if $j \notin I(B)$. Clearly, if $j \in I(B)$, then the respective probability is computed for one fixed value v_j.

3. An illustrative example

The somewhat complicated procedure for computing FPs, based on Theorem 1, will now be illustrated on the example of a biogas production unit. The BN for this unit's safety analysis is presented in Malinowski 2025. Fig. 1 shows a fragment of this BN, extracted from the complete network to demonstrate the calculations performed using Eqs. (6)-(10). Random variables are assigned to the safety-related events specified in Fig. 1 as follows:

X_1 – digester operation mode
X_2 – high level sensor failure
X_3 – ice in the outlet gas pipe
X_4 – outlet gas valve closed
X_5 – over/under pressure protection failure
X_6 – foam formation
X_7 – rising substrate level
X_8 – outlet gas pipe blocked
X_9 – digester overflow
X_{10} – overpressure in digester
X_{11} – substrate spill
X_{12} – digester rupture
X_{13} – gas dispersion
X_{14} – poisoning or asphyxiation
X_{15} – external explosion or fire

The above random variables are assumed to take binary values, e.g. X_2=1/0 if the high level sensor is failed/in operating condition.

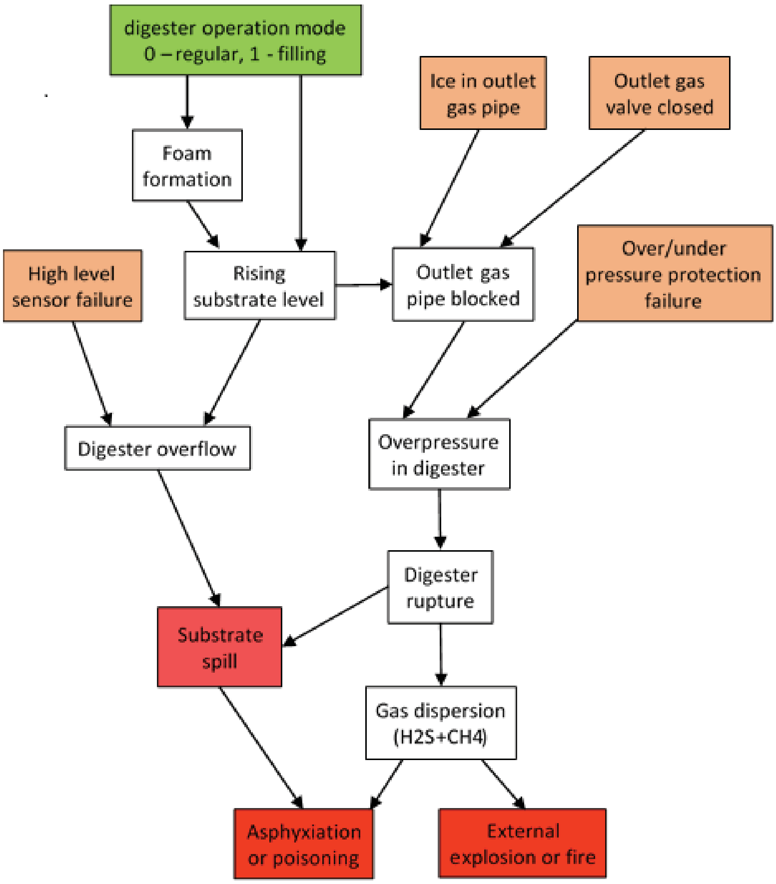

Fig. 1. Fragment of BN for biogas production unit

Let $A=\{X_2, X_3, X_4\}$ and $B=\{X_{11}, X_{15}\}$ be the sets of the causing and resulting events, respectively. From the definitions in the previous section, we obtain:

$U(B)=U_0(B)\cup\ldots\cup U_4(B)$, where

$U_0(B)=\{X_1, X_2, X_3, X_4, X_5\}$,

$U_1(B)=\{X_6, X_7, X_8, X_9, X_{10}\}$, $W_1(B)=\{X_9, X_{10}\}$,

$U_2(B)=W_2(B)=\{X_{11}, X_{12}\}$,

$U_3(B)=W_3(B)=\{X_{13}\}$,

$U_4(B)=\{X_{15}\}$, $W_4(\mathrm{B})=\varnothing$.

Hence, the number of layers $k(B)$ equals 5. Note that nodes X_9 and X_{10} are lower border nodes in layer 1, which comprises five nodes. Also note that $U_0(B)\backslash A=\{X_1, X_5\}$, meaning that there can be initial nodes that do not belong to A.

Let us compute the successive R_q, q=1,…,4. To shorten the notation, $X_j=v_j$ will be abbreviated to v_j if the complete expression is too long. R_1 is computed using Eq. (6). We have:

$J_1(B) = \{9{,}10\}$,

$[I_0(B) \cup I_1(B)]\backslash[I(A \cup B) \cup J_1(B)] =$

$= \{1, \ldots, 10\}\backslash\{2{,}3{,}4{,}11{,}15{,}9{,}10\}$

$= \{1{,}5{,}6{,}7{,}8\}$,

$I_1(B) = \{6{,}7{,}8{,}9{,}10\}$,

$$I_0(B)\backslash I(A) = \{1{,}5\} \tag{11}$$

Substituting these sets in Eq. (6), we obtain:

$$R_1(v_9, v_{10}) = \sum_{\substack{v_j \in V(j),\\ j\in\{1,5,6,7,8\}}} \downarrow$$

$$\prod_{i\in\{6,7,8,9,10\}} \Pr[v_i \mid v_h,\ h \in P(i)] \times \times \prod_{i\in\{1,5\}} \Pr(v_i)$$

$$= \sum_{\substack{v_1\in V(1),\\ v_5\in V(5),\\ v_6\in V(6),\\ v_7\in V(7),\\ v_8\in V(8)}} \begin{matrix} \Pr(v_6 \mid v_1) \times \\ \Pr(v_7 \mid v_1, v_6) \times \\ \Pr(v_8 \mid v_3, v_4, v_7) \times \\ \Pr(v_9 \mid v_2, v_7) \times \\ \Pr(v_{10} \mid v_5, v_8, v_9) \times \\ \times \Pr(v_1)\Pr(v_5) \end{matrix} \quad (12)$$

To compute R_2 and R_3, we use Eq. (7) with q=1 and q=2, respectively. Note that

$J_2(B) = \{11,12\}$,

$[J_1(B) \cup I_2(B)]\backslash[I(B) \cup J_2(B)] =$

$= \{9,10,11,12\}\backslash\{11,15,12\} = \{9,10\}$,

$I_2(B) = \{11,12\}$,

$$J_1(B) = \{9,10\} \quad (13)$$

Therefore

$R_2(v_{11}, v_{12}) =$

$$= \sum_{\substack{v_j\in V(j),\\ j\in\{9,10\}}} \begin{matrix} \prod_{i\in\{11,12\}} \Pr[v_i \mid v_h,\ h \in P(i)] \times \\ \times R_1(v_9, v_{10}) \end{matrix}$$

$$= \sum_{\substack{v_9\in V(9),\\ v_{10}\in V(10)}} \begin{matrix} \Pr(v_{11} \mid v_9, v_{12}) \times \\ \Pr(v_{12} \mid v_{10}) \times \\ \times R_1(v_9, v_{10}) \end{matrix} \quad (14)$$

Further, we have:

$J_3(B) = \{13\}$,

$[J_2(B) \cup I_3(B)]\backslash[I(B) \cup J_3(B)] =$

$= \{11,12,13\}\backslash\{11,15,13\} = \{12\}$,

$I_3(B) = \{13\}$,

$$J_2(B) = \{11,12\} \quad (15)$$

Thus

$R_3(v_{13}) =$

$$= \sum_{\substack{v_j\in V(j),\\ j\in\{12\}}} \begin{matrix} \prod_{i\in\{13\}} \Pr[v_i \mid v_h,\ h \in P(i)] \times \\ \times R_2(v_{11}, v_{12}) \end{matrix}$$

$$= \sum_{v_{12}\in V(12)} \begin{matrix} \Pr(v_{13} \mid v_{12}) \times \\ \times R_2(v_{11}, v_{12}) \end{matrix} \quad (16)$$

In the last step, we compute R_4 from Eq. (8). The following equalities hold:

$I_4(B) \cap I(B) = \{15\} \cap \{11,15\} = \{15\}$,

$[J_3(B) \cup I_4(B)]\backslash I(B) =$

$= \{13,15\}\backslash\{11,15\} = \{13\}$,

$I_4(B) = \{15\}$,

$$J_3(B) = \{13\} \quad (17)$$

Hence, we obtain:

$R_4(v_{15}) =$

$$= \sum_{\substack{v_j\in V(j),\\ j\in\{13\}}} \begin{matrix} \prod_{i\in\{15\}} \Pr[v_i \mid v_h,\ h \in P(i)] \times \\ \times R_3(v_{13}) \end{matrix}$$

$$= \sum_{v_{13}\in V(13)} \begin{matrix} \Pr(v_{15} \mid v_{13}) \times \\ \times R_3(v_{13}) \end{matrix} \quad (18)$$

As follows from Theorem 1, the sought forward probability is equal to $R_4(v_{15})$, i.e.

$\Pr\left[X_j = v_j,\ j \in I(B) \mid X_j = v_j,\ j \in I(A)\right] =$

$$= \Pr\left(\begin{matrix} X_{11} = v_{11}, \\ X_{15} = v_{15} \end{matrix} \middle| \begin{matrix} X_2 = v_2, \\ X_3 = v_3, \\ X_4 = v_4 \end{matrix} \right) = R_4(v_{15}) \quad (19)$$

4. Possible applications of FPs and RPs

As mentioned in the introduction, FPs and RPs are necessary for quantitative cause-effect analysis of various event configurations in BN-modeled systems. Referring to the example in the previous section, we can, for instance, compute the probability of substrate spill in the absence of explosion or fire, provided that the high level sensor is in operating state, and there is ice in the outlet gas pipe or the outlet gas valve is closed. It is easy to check that this probability is equal to

$$\frac{\sum_{i=1}^{3} \Pr(X_{11} = 1, X_{15} = 0 \mid E_i)\Pr(E_i)}{\sum_{i=1}^{3} \Pr(E_i)} \quad (20)$$

where

$E_1 = \{X_2 = 0, X_3 = 0, X_4 = 1\}$,

$E_2 = \{X_2 = 0, X_3 = 1, X_4 = 0\}$,

$$E_3 = \{X_2 = 0, X_3 = 1, X_4 = 1\} \quad (21)$$

The conditional probabilities in the numerator in Eq. (20) can be calculated from Eq. (19). The probabilities $\Pr(E_i)$ are easy to compute, because E_1, E_2, E_3 are intersections of independent initial events, e.g.

$$\Pr(E_1)=\Pr(X_2=0)\Pr(X_3=0)\Pr(X_4=1) \quad (22)$$

A reverse probability is needed, for example, when we want to compute the probability that ice in the outlet gas pipe is a likely cause of the occurred substrate spill. This probability is equal to $\Pr(X_3=1 \mid X_{11}=1)$. As follows from Eqs. (3) and (5), we have:

$$\Pr(X_3 = 1 \mid X_{11} = 1) = = \Pr(X_{11} = 1 \mid X_3 = 1)\frac{\Pr(X_3=1)}{\Pr(X_{11}=1)} \quad (23)$$

where

$$\Pr(X_{11} = 1) = = \Pr(X_{11} = 1 \mid X_3 = 0)\Pr(X_3 = 0) + + \Pr(X_{11} = 1 \mid X_3 = 1)\Pr(X_3 = 1) \quad (24)$$

Thus, to compute $\Pr(X_3=1 \mid X_{11}=1)$ we need the probabilities of the initial events $\{X_3=1\}$ and $\{X_3=0\}$, and also the FPs $\Pr(X_{11}=1 \mid X_3=0)$ and $\Pr(X_{11}=1 \mid X_3=1)$ that can be found using Theorem 1.

As shown above, appropriately defined FPs and RPs allow for a detailed probabilistic analysis of cause-effect relations in a system modeled by a BN.

5. Concluding remarks and future work

The author presents a novel method for inference in Bayes networks that computes forward and reverse conditional probabilities with a recursive procedure. As demonstrated in the preceding section, these probabilities can be useful in a detailed analysis of cause-effect relations in BN-modeled systems. FPs and RPs can be regarded as „extended” conditional probability tables (CPTs), by analogy to usual CPTs relating individual nodes in a BN to their parents. Note that FP relates multiple non-initial nodes to multiple initial ones, while RP specifies the reverse relation.

The number of summations in one step of the recursive procedure using Eqs. (6)-(10) can be significantly less than the number of summations in Eq. (3), thus reducing the computational effort. This effect is particularly noticeable if the width of a BN (i.e. the maximum number of nodes in one layer) is small compared to the total number of its nodes. Besides, if $k(B)\geq 3$, then, in addition to the target FP (i.e. $R_{k(B)-2}$), the recursive method computes intermediate FPs (i.e. $R_1,\ldots,R_{k(B)-2}$) that can also be useful for the BN analysis.

Admittedly, even the recursive approach is computationally costly for a BN with a complex structure and/or multi-valued variables. However, the computational effort can be acceptable for a BN with binary variables, such as the one considered in Section 3. This is because, in the binary case, the sums in Eqs. (6)-(8) and (10) have at most 2^w components, where w is roughly equal to the BN width (check the indexes of summation in these formulas). If possible, it is advisable to replace multi-valued variables with appropriate configurations of variables with fewer values. For example, the random variable “precipitation” with the following five values: 0 – no precipitation, 1 – light rain, 2 – heavy rain, 3 – light snow, 4 – heavy snow, can be replaced by two three-valued variables “rain” and “snow”. Such replacements increase the number of nodes in a BN, leading to smaller CPTs, which can improve computational efficiency.

Future work will be focused on two objectives. The first is to develop a procedure for calculating FPs with non-initial conditioning events, i.e. the probabilities $\Pr[X_j=v_j, j\in I(B) \mid X_j=v_j, j\in I(A)]$, where A can include non-initial nodes. The second is to examine the feasibility of transforming a BN with multi-valued variables into a BN whose variables have fewer values, such as binary or unary. This kind of transformation, combined with a recursive approach to the FPs computation, can enhance the efficiency of quantitative analysis of BNs.

References

Ding, J. (2010). Probabilistic Inferences in Bayesian Networks. *ArXiv*. https://arxiv.org/abs/1011.0935

Gansch, R., L. Putze, T. Koopmann, J. Reich and C. Neurohr (2025). Causal Bayesian Networks for Data-driven Safety Analysis of Complex Systems. *ArXiv*. https://arxiv.org/abs/2505.19860

Heckerman, D. (2022). A Tutorial on Learning With Bayesian Networks. arXiv:2002.00269v3. https://arxiv.org/abs/2002.00269

Jensen, F.V. and T.D. Nielsen (2007). *Bayesian Networks and Decision Graphs. Second Edition*. Springer.

Kjaerluff, U.B. and A.L. Madsen (2013). *Bayesian Networks and Influence Diagrams: A Guide to Construction and Analysis. Second Edition*. Springer.

Lu, Y., Q. Zheng and D. Quinn (2023). Introducing Causal Inference Using Bayesian Networks and do-Calculus. *Journal of Statistics and Data Science Education 31*, 3–17.

Malinowski, J. (2025). A Safety Model for Industrial Environment Based on the Bayesian Network Paradigm. In E.B. Abrahamsen, T. Aven, F. Bouder, R. Flage and M. Ylönen (Eds.), *Proceedings of the 35th European Safety and Reliability Conference (ESREL 2025) and the 33rd Society for Risk Analysis Europe Conference (SRA-E 2025)*, pp. 1240-1247. Research Publishing.

Scutari, M. and J.-M. Denis (2021). *Bayesian Networks with Examples in R*. Chapman and Hall/CRC.

Tchórzewska-Cieślak, B., K. Pietrucha-Urbanik and D. Szpak (2017). Development of cause-effect dependence model of undesirable events using Bayes network. *Journal of Polish Safety and Reliability Association 8*, 149–156.

Vázquez-Aguirre, J.B., N. Cruz-Ramírez and M. Quiroz-Castellanos (2022). A Model for Learning Cause-Effect Relationships in Bayesian Networks. In: Castillo, O. and P. Melin (Eds.), *New Perspectives on Hybrid Intelligent System Design based on Fuzzy Logic, Neural Networks and Metaheuristics. Studies in Computational Intelligence 1050*. Springer, Cham. https://doi.org/10.1007/978-3-031-08266-5_25

Wang, Q.-A., J. Chen, Y.-Q. Ni., Y.-F. Xiao, N.-B. Liu, S.-K. Liu and W.-S. Feng (2025). Application of Bayesian networks in reliability assessment: A systematic literature review. *Structures 71 108089*.