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A Prescriptive Maintenance Policy for Systems with a Bathtub-Shaped Degradation Rate

Antonio Piscopo
Scuola Superiore Meridionale, Italy; Université d'Angers, France. E-mail: antonio.piscopo@unina.it

Bruno Castanier
Université d'Angers, France. E-mail: bruno.castanier@univ-angers.fr

Mitra Fouladirad
Aix Marseille Université, France. E-mail: mitra.fouladirad@centrale-marseille.fr

Massimiliano Giorgio
Università di Napoli Federico II, Italy. E-mail: massimiliano.giorgio@unina.it

In this paper, we propose a prescriptive maintenance policy for systems whose degradation evolves according to a non-homogeneous gamma process, where the degradation rate function (here intended as the derivative of the mean degradation function) has a bathtub-shaped temporal evolution. We introduce a usage variable that represents the rate at which the system operates, with a nominal level corresponding to standard operating conditions. At each inspection, this rate can be reduced by selecting one level within a finite set of admissible ones. The usage rate directly influences the degradation rate: lower usage rates slow down the degradation process, but each reduction involves a cost that increases with its deviation from the nominal level and with the duration for which that reduction is applied. In this work, the system operates over a finite time horizon and is subject to periodic inspections that are perfect, instantaneous, and non-destructive. Failures are not self-announcing. At each inspection, the usage rate to be applied until the next inspection is selected, with the possibility of performing preventive or corrective maintenance. The optimal policy is obtained by minimizing the expected total cost over the finite horizon.
Numerical studies are carried out by varying the number of available usage-rate levels and the length of the operating horizon. The obtained results, compared to those obtained by adopting a similar (simpler) policy that allows for corrective and preventive replacements only, show that within the considered experimental scenarios, allowing for usage-rate reductions significantly reduces the expected cost.

Keywords: prescriptive maintenance, non-homogeneous gamma process, bathtub-shaped degradation rate, finite-horizon maintenance policy, average total cost.

1. Introduction

Prescriptive Maintenance (PsM) (Meissner et al. 2021, Longhitano et al. 2022) has recently emerged as a paradigm that goes beyond predictive maintenance (PdM) approaches by recommending optimal actions that not only include traditional maintenance interventions, such as inspections or corrective and preventive replacements, but also prescribe how the system should be operated.

Examples of PsM are given in Esposito et al. (2022) and Dachraoui et al. (2025), where the prescriptive action consists of adjusting the usage-rate of the system to accelerate and/or decelerate the degradation process. Reducing the usage rate reduces the risk of failure and the associated (expected) costs. However, it results in a loss of performance, which in turn causes additional costs. The optimal decision is determined by finding the best trade-off between these effects. Both these papers assume that the degradation process is a homogeneous gamma process.

By following Esposito et al. (2022) and Dachraoui et al. (2025), in this paper we propose a policy that, in addition to classical maintenance actions, includes the possibility of tuning the usage rate whenever it is beneficial.

The main novelty with respect to this recent literature is that here the degradation process is modeled using a gamma process with bathtub-shaped degradation rate (DR) function, intended as the derivative of the mean function. This characterization of the gamma process, first proposed in Giorgio et al. (2024), allows the description of situations where, as it is often observed in real-world degrading systems (Gertsbakh and Kordonskiy 1969), the degradation process evolves in three phases: an early phase where the DR decreases, a second phase where it is approximately constant and a final one where it increases. In these cases using a suitable degradation model is of paramount importance, because the misspecification of the DR function may yield suboptimal decisions and higher expected costs (Esposito et al. 2024; Sun et al. 2023).

The system is considered failed when its degradation level exceeds a critical threshold. It is assumed that failures are not self-announcing and that a failed system can continue to operate, albeit this incurs additional costs (Bismut et al. 2022).

The proposed PsM policy assumes that maintenance actions can be made only at inspection epochs. Available actions consist of performing a preventive maintenance (PM), a corrective maintenance (CM), adjusting the usage rate, and/or leaving the unit in operation until the next inspection (i.e., do not perform any maintenance action). Both PM and CM are assumed to restore the system to its as-good-as-new state. The effect of the usage-rate on the degradation growth is modeled by using an appropriate link function.

The problem of determining the optimal policy (i.e., the one that minimizes the total expected maintenance cost) is defined over a finite time horizon, which is common in industrial environments where maintenance planning is constrained by production campaigns, contractual service periods, or insurance deadlines (Nakagawa and Mizutani 2009, Ponchet et al. 2011, Sánchez-Herguedas et al. 2024).

The rest of the paper is organized as follows. Sections 2 introduces the non-homogeneous gamma process with bathtub-shaped DR and describes how its evolution is influenced by the usage-rate. Section 3 defines the cost structure, the maintenance policy, and formulates the problem as a finite-horizon Markov decision process (MDP) with periodic inspections. Sections 4 derives the optimal policy by discretizing the degradation state and applying a backward recursion algorithm (BRA) (Bertsekas 2012). Section 5 reports a numerical study that highlights how the proposed policy adapts to the bathtub-shaped DR. Section 6 concludes the paper.

2. The Degradation Model

The non-homogeneous gamma process, $\{Z(t), t \geq 0\}$, is a process with independent increments. Hence, given an initial condition ($Z(0) = 0$), it is fully characterized by the distribution of its generic increment $\Delta Z(t,s) = Z(t+s) - Z(t)$ $(s > 0)$. Specifically, the probability density function (pdf) and the cumulative distribution function (Cdf) of $\Delta Z(t,s)$ can be formulated as:

$$f_{\Delta Z(t,s)}(\delta) = \frac{\lambda^{\Delta\eta(t,s)}\, \delta^{\Delta\eta(t,s)-1}\, e^{-\lambda\delta}}{\Gamma\big(\Delta\eta(t,s)\big)} \quad (1)$$

and

$$F_{\Delta Z(t,s)}(\delta) = \frac{\gamma(\Delta\eta(t,s), \lambda\,\delta)}{\Gamma(\Delta\eta(t,s))}, \quad (2)$$

respectively, where $\Gamma(\cdot)$ is the gamma function and $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function. $\lambda > 0$ is the scale parameter, $\Delta\eta(t,s) = \eta(t+s) - \eta(t)$, and $\eta(t)$ is the shape parameter that is a non-decreasing function with $\eta(0) = 0$.

The mean of $Z(t)$ is given by

$$\mathrm{E}\{Z(t)\} = \eta(t)/\lambda. \quad (3)$$

By definition, the DR function is the derivative, with respect to t, of the mean degradation function in Eq. (3):

$$\frac{\mathrm{d}}{\mathrm{d}t} E\{Z(t)\} = \frac{1}{\lambda}\left[\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)\right]. \quad (4)$$

Following Giorgio et al. (2024), to obtain a bathtub-shaped DR, we express $\eta(t)$ as:

$$\eta(t) = a_1 t^{b_1} + a_2 t^{b_2}, \quad (5)$$

with $a_1, a_2, b_1, b_2 > 0$, yielding the DR function:

$$\frac{\mathrm{d}}{\mathrm{d}t} E\{Z(t)\} = \frac{a_1 b_1\, t^{b_1-1}}{\lambda} + \frac{a_2 b_2\, t^{b_2-1}}{\lambda}. \quad (6)$$

This function is bathtub-shaped when $(b_1 - 1)/(b_2 - 1) < 0$ (Giorgio et al. 2024).

We assume that the system is failed when its degradation level exceeds an assigned critical threshold, say l_F. The useful life X of the system (that is, its time to failure) is defined as the first passage time of the degradation process to l_F (i.e., $X = \inf\{x{:}\, Z(x) > l_F\}$). Consequently, since $Z(t)$ is a non-decreasing process, from Eq. (2), for any $x > 0$, the Cdf of X can be formulated as

$$F_X(x) = 1 - F_{\Delta Z(0,x)}(l_F), \quad (7)$$

where $\Delta Z(0, x) = Z(x)$.

From Eq. (7), the expected value of the useful life is:

$$E\{X\} = \int_0^{+\infty} \left[1 - F_{\Delta Z(0,x)}(l_F)\right] \mathrm{d}x. \quad (8)$$

The remaining useful life (RUL) of a unit at t is defined as $X_t = \max(X - t, 0)$. Since the process has independent increments, the conditional Cdf of X_t, given $Z(t) = z$, $z \le l_F$, is:

$$F_{X_t|Z(t)}(x_t|z) = 1 - F_{\Delta Z(t,x_t)}(l_F - z), \quad (9)$$

with $x_t > 0$. Moreover, for any $s > 0$, the conditional expected value of $\max(s - X_t, 0)$ given $Z(t) = z$ is:

$$E\{\max(s - X_t, 0)|z\} = \int_0^{s} [1 - F_{\Delta Z(t,x_t)}(l_F - z)]\, \mathrm{d}x_t\,, \quad (10).$$

This latter result will be used in Section 3.

2.1. *Effect of Varying the Usage-Rate on the Degradation Process*

In this work, we assume that the system can operate at any of a finite number, n_u, of usage-rate levels, $u_1, u_2, \dots, u_{n_u}$, where u_1 is the nominal usage-rate and the remaining levels are listed in decreasing order. Higher-than-nominal usage-rates are not allowed.

We assume that, under the usage-rate u_k, the degradation is still described by a non-homogeneous gamma process and that the usage-rate level only affects its shape function, which is expressed as:

$$\eta_k(t) = \alpha_{1,k}\, a_1 t^{b_1} + \alpha_{2,k}\, a_2 t^{b_2}, \quad (11)$$

where $\alpha_{1,1} = \alpha_{2,1} = 1$ and both $\alpha_{1,k}$ and $\alpha_{2,k}$ decrease with k (i.e., as the usage rate decreases).

It is also assumed that degradation increments over disjoint intervals are conditionally independent given the selected usage-rate levels. Hereinafter, for notational convenience, we will denote by $\Delta Y_k(t, s)$ the degradation accumulated by the system over the interval $(t, t + s)$ under the usage-rate u_k.

3. Maintenance Policy and Cost Model

Let T be the finite time horizon during which the system operates. We assume that $n - 1$ inspections are performed at $n - 1$ predetermined inspection times $t_1, \dots, t_{n-1}$, which, for simplicity, yet without loss of generality, are assumed to be equispaced, with $t_j = j\,\tau$, $j = 1,\dots,n - 1$ and $\tau = T/n$. It is assumed that inspections are instantaneous and not destructive. A decision about the usage rate to adopt up to the first inspection is also made at time t_0, where no inspection takes place. Similarly, no inspection takes place at $T = t_n$, where the system is dismissed regardless of its state, at zero cost.

Let $W(t_j^-)$ and $W(t_j^+)$ denote the degradation level of the system immediately before and after the $j-th$ inspection, respectively, and let w_j^- and w_j^+ be their corresponding realizations.

At t_j, a pair of actions $\boldsymbol{A}_j = (m, u)_j$ is selected, where m denotes the maintenance action performed at time t_j and u is the usage-rate level to be applied up to the next inspection time $t_{j+1} = t_j + \tau$. Available options for m are PM, CM, or no replacement (indicated as action 0). Options for u are $u_1, u_2 \dots, u_{n_u}$.

Hence, the action set is $\mathcal{A} = \{(\mathrm{CM}, u_1), \dots, (\mathrm{CM}, u_{n_u}), (\mathrm{PM}, u_1), \dots, (\mathrm{PM}, u_{n_u}), (0, u_1), \dots, (0, u_{n_u})\}$, which contains $3 \times n_u$ options.

Since both CM and PM restore the system to its as-good-as-new state, after selecting $\boldsymbol{A}_j = (m, u)_j$, the degradation level is updated as

$$w_j^+ = w_j^-\, I_{\{m=0\}} \quad (12)$$

where $I_{\{B\}}$ denotes the indicator function and equals 1 if the statement B is true and 0 otherwise.

Let V_j^- denote the system age immediately before the inspection at time t_j, defined as the time elapsed since the last replacement, and let v_j^- be its realization. The use of the capital letter for V_j^- serves to remark that it is a random variable. In fact, its value depends on past actions, which in turn depend on past observations of the degradation process. Given $\boldsymbol{A}_j$ and $V_j^- = v_j^-$, the system age is updated via the following recursive equations:

$$v_j^+ = v_j^-\, I_{\{m=0\}} \quad (13)$$

$$v_{j+1}^- = v_j^+ + \tau \quad (14)$$

where v_j^+ is the system age immediately after the maintenance action at t_j. Obviously, by definition, it will be $v_j^+ \le v_j^- \le t_j$ and $v_0^- = v_0^+ = 0$. V_j^- takes values in the discrete set $\{0, t_1, \dots, t_j\}$. To ensure the Markov property of the decision process, we define the state of the system at time t_j as the pair $\left(W\left(t_j^-\right), V_j^-\right)$, whose realization is $\left(w_j^-, v_j^-\right)$.

The expected cost $\mathcal{C}_j\left(w_j^-, v_j^-; \boldsymbol{A_j}\right)$ incurred over the interval $\left(t_j, t_{j+1}\right)$, which depends on $\left(w_j^-, v_j^-\right)$ and $\boldsymbol{A}_j$, accounts for the following contributions:

i) the inspection cost (not paid for $j = 0$),
ii) the costs associated with the selected maintenance action m,
iii) the usage cost due to the selected usage-rate level u,
iv) the expected downtime cost, which depends on the RUL at time t_j.

Once $\boldsymbol{A}_j$ is selected, w_j^+ and v_j^+ are determined via Eqs. (12) and (13). Let $X_{v_j^+}$ denote the RUL at time t_j given that the system age is v_j^+. The downtime cost over the interval $\left(t_j, t_{j+1}^-\right)$ is defined as:

$$\mathcal{C}_D\left(X_{v_j^+}\right) = c_d \max\left\{\tau - X_{v_j^+}, 0\right\},$$

where c_d is the downtime cost rate.

Using Eq. (10), the conditional expected downtime cost, given $W\left(t_j^-\right) = w_j^-, V_j^- = v_j^-$ and $\boldsymbol{A}_j$, can be written as:

$$E\left\{\mathcal{C}_D\left(X_{v_j^+}\right)\middle| w_j^-, v_j^-; \boldsymbol{A}_j\right\} = c_d \int_0^{\tau} \left[1 - F_{\Delta Y_k\left(v_j^+, x\right)}\left(l_F - \mathrm{w}_j^+\right)\right] \mathrm{d}x. \quad (15)$$

Hence, by adding these contributions, it is:

$$\mathcal{C}_j\left(w_j^-, v_j^-; \boldsymbol{A}_j\right) = E\left\{\mathcal{C}_D\left(X_{v_j^+}\right)\middle| w_j^-, v_j^-; \boldsymbol{A}_j\right\} + c_i + c_p \mathrm{I}_{\{m=\mathrm{PM}\}} + c_c \mathrm{I}_{\{m=\mathrm{CM}\}} + \sum_{k=1}^{n_u} c_{u_k} \tau\, \mathrm{I}_{\{u=u_k\}} \quad (16)$$

where c_i is the inspection cost, c_p is the PM cost, c_c is the CM cost, and c_{u_k} is the cost per unit of time for operating the system with the usage-rate level u_k.

A maintenance policy $\boldsymbol{\pi} = \{\pi_0, \pi_1, \dots, \pi_{n-1}\}$ is a collection of decision rules (i.e., one for each j= 0, 1, …, $n - 1$) such that, at time t_j, π_j selects a pair of actions $\boldsymbol{A}_j$ based on the observed state $\left(w_j^-, v_j^-\right)$ (Bertsekas 2012). Thus, since π_j is a function of $\left(w_j^-, v_j^-\right)$, the decision rule applied at t_j can be more formally defined as:

$$\boldsymbol{A_j} = \pi_j\left(w_j^-, v_j^-\right). \quad (17)$$

The total expected maintenance cost over the horizon T, under the policy $\boldsymbol{\pi}$, can therefore be written as:

$$E\{\mathcal{C}_T(\boldsymbol{\pi})\} = \sum_{j=0}^{n-1} E\left\{\mathcal{C}_j\left(W\left(t_j^-\right), V_j^-; \boldsymbol{A}_j\right)\right\}, \quad (18)$$

where $\boldsymbol{A}_j$ is determined by applying the decision rule (17).

4. Optimal Maintenance Policy

Our goal is to find an optimal policy $\boldsymbol{\pi}^*$ that minimizes the total expected maintenance cost over the horizon T in Eq. (18). Given the formulation in Section 3, the decision problem is a finite-horizon MDP, and the assumptions required to formulate the Bellman equation are satisfied. Therefore, the optimization problem can be solved by applying the BRA (Bertsekas 2012).

Let $\mathcal{J}_j\left(w_j^-, v_j^-\right)$ denote the Bellman cost-to-go function at the inspection time t_j, representing the minimum expected cost from t_j to the horizon T, when the system is in state $\left(w_j^-, v_j^-\right)$.

At the terminal condition $j = n$, we set, for any possible state (w_n^-, v_n^-),

$$\mathcal{J}_n(w_n^-, v_n^-) = 0, \quad (19)$$

for $j = n - 1, \dots, 0$, BRA proceeds by computing $\mathcal{J}_j\left(w_j^-, v_j^-\right)$ via the Bellman recursion:

$$\mathcal{J}_j\left(w_j^-, v_j^-\right) = \min_{A_j \in \mathcal{A}}\left\{\mathcal{C}_j\left(w_j^-, v_j^-; \boldsymbol{A}_j\right) = E\left\{\mathcal{J}_{j+1}\left(W_{j+1}^-, V_{j+1}^-\right) | w_j^-, v_j^-; \boldsymbol{A}_j\right\}\right\} \quad (20)$$

The expectation in Eq. (20) can be evaluated as:

$$E\left\{\mathcal{J}_{j+1}\left(W_{j+1}^-, V_{j+1}^-\right) | w_j^-, v_j^-; \boldsymbol{A}_j\right\} = \int_0^{+\infty} \mathcal{J}_{j+1}\left(w_j^+ + y, v_{j+1}^-\right) f_{\Delta Y_k(t_j, \tau)}(y) \mathrm{d}y,$$

where w_j^+ and v_{j+1}^- are given by Eqs. (12)-(14).

Accordingly, each decision rule of the optimal policy $\boldsymbol{\pi}^* = \{\pi_0^*, \pi_1^*, \dots, \pi_{n-1}^*\}$ is defined as follows. For each inspection time t_j and state $\left(w_j^-, v_j^-\right)$, the optimal decision rule $\pi_j^*\left(w_j^-, v_j^-\right)$ selects a pair of actions $\boldsymbol{A}_j$ from the action set $\mathcal{A}$ that achieves the minimum in Eq. (20).

It is worth noting that, by definition, $\mathcal{J}_0(0, 0)$ equals the minimum total expected maintenance cost over the horizon T in Eq. (18):

$$E\{C_T(\boldsymbol{\pi}^*)\} = \mathcal{J}_0(0,0). \tag{21}$$

Because the degradation component of the state is continuous, the Bellman equation does not admit a closed-form solution. We therefore obtain a numerical approximation by discretizing the state space, as described in the next section.

4.1. Discretization of the State Space

The continuous degradation space $(0, +\infty)$ is discretized using $N+2$ representative values.

Let $g_0, g_1, \ldots, g_{N-1}, g_N$ be equally spaced partition points on $(0, l_F)$, with $g_i = i\, l_F/N$, $i = 0, \ldots, N$ and let $\Delta g = l_F/(2N)$. Based on these quantities, Table 1 describes the rule to replace w_j^- and w_j^+ by the representative values s_j^- and s_j^+ in the discrete state-space model. Accordingly, the discretized state at t_j is the pair (s_j^-, v_j^-), approximating the continuous state (w_j^-, v_j^-). For $j = 0$, the only possible state is $(0,0)$. For $j > 0$, the number of admissible states is at most $N(j+1)$. Under this discretization, the cost expressions in Eq. (16) retain the same analytical form, with continuous degradation levels (w_j^- and w_j^+) approximated by their representative values (s_j^- and s_j^+).

Table 1. Rule for transforming $w_j^\#$ into $s_j^\#$ (where # is either $-$ or $+$).

$w_j^\#$	$s_j^\#$
$w_j^\# = 0$	$s_j^\# = 0$
$\vdots$	$\vdots$
$g_i < w_j^\# \le g_{i+1}$	$s_j^\# = g_i + \Delta g$
$\vdots$	$\vdots$
$w_j^\# > g_N = l_F$	$s_j^\# = g_N = l_F$

Expectations in Eq. (20) can be approximated as follows. Let $q_j\big(s_{j+1}^- | s_j^+, v_j^+; \boldsymbol{A}_j\big)$ denote the conditional probability that at time t_{j+1} the degradation level is represented by s_{j+1}^- given $\big(s_j^+, v_j^+\big)$ and $\boldsymbol{A}_j$.

By the monotonicity of the gamma process, degradation cannot decrease, thus, if $s_{j+1}^- < s_j^+$,

$$q_j\big(s_{j+1}^- | s_j^+, v_j^+; \boldsymbol{A}_j\big) = 0.$$

If $s_j^+ \le s_{j+1}^- < l_F$, define

$$l_l = s_{j+1}^- - \Delta g - s_j^+, \ \ l_u = s_{j+1}^- + \Delta g - s_j^+,$$

then

$$q_j\big(s_{j+1}^- | s_j^+, v_j^+; \boldsymbol{A}_j\big) = F_{\Delta Y_k(v_j^+, \tau)}(l_u) - F_{\Delta Y_k(v_j^+, \tau)}(l_l).$$

Finally, if $s_{j+1}^- = l_F$,

$$q_j\big(s_{j+1}^- | s_j^+, v_j^+; \boldsymbol{A}_j\big) = 1 - F_{\Delta Y_k(v_j^+, \tau)}\big(l_F - s_j^+\big).$$

Given the discretization introduced above, the Bellman equation at t_j is defined over the discrete state (s_j^-, v_j^-).

For $j = n$, for any possible state (s_n^-, v_n^-), it is:

$$\mathcal{J}_n(s_n^-, v_n^-) = 0, \tag{22}$$

whereas, for $j = n-1, \ldots, 0$, the Bellman equation is:

$$\mathcal{J}_j\big(s_j^-, v_j^-\big) = \min_{\boldsymbol{A}_j \in \mathcal{A}} \Big\{ C_j\big(s_j^-, v_j^-; \boldsymbol{A}_j\big) + \sum_{s_{j+1}^- \ge s_j^+} \mathcal{J}_{j+1}\big(s_{j+1}^-, v_{j+1}^-\big)\, q_j\big(s_{j+1}^- | s_j^+, v_j^+; \boldsymbol{A}_j\big) \Big\} \tag{23}$$

After this discretization, the optimal policy $\boldsymbol{\pi}^*$ can be obtained by applying the BRA (Bertsekas 2012).

5. Numerical Analysis

The numerical analysis is performed using the following degradation parameters: $a_1 = 10$, $a_2 = 10^{-3}$, $b_1 = 0.3$, $b_2 = 4$ and $\lambda = 1$, with failure threshold $l_F = 30$. These parameter values satisfy the condition under which the DR function is bathtub-shaped (see Section 2).

We consider three different time horizons T_1, T_2, and T_3. T_1 is set to 6, a value that is close to the expected lifetime of the system operating at the nominal usage-rate, which, from Eq. (8), equals 5.97. During this time horizon, $n_1 - 1 = 11$ periodic inspections are performed, so that the inspection interval is $\tau = 0.5$. Keeping the same inspection interval, the other two time horizons are obtained by setting $n_2 - 1 = 12$ and $n_3 - 1 = 13$ inspections, respectively. Accordingly, $T_2 = 6.5$ and $T_3 = 7$. Under the nominal usage-rate, these values correspond approximately to the 82^{nd} and 98^{th} percentiles of the lifetime computed via Eq. (7).

Regarding the effect of the usage-rate, for all $k = 1, \ldots, n_u$, we assume that selecting a certain u_k affects the two coefficients $\alpha_{1,k}$ and $\alpha_{2,k}$ of the shape function in Eq. (11) in the same way, i.e., $\alpha_{1,k} = \alpha_{2,k} = \alpha_k$. Accordingly, four usage-rate levels u_1, u_2, u_3, and u_4 are considered, with corresponding values of α_k:

$\alpha_1 = 1$, $\alpha_2 = 0.75$, $\alpha_3 = 0.5$, and $\alpha_4 = 0.25$, where α_1 corresponds to the nominal usage-rate.

Based on these usage-rate levels, three different setups $\mathcal{S}_K$, with $K = 1, 2, 3$, are defined, each allowing a different number of usage-rate levels. Specifically, in the setup $\mathcal{S}_1$, only the nominal usage-rate u_1 is available, thus the corresponding policy reduces to a simpler PdM policy. In the case of the setup $\mathcal{S}_2$, the choice is between u_1 and u_3, whereas, in the case of $\mathcal{S}_3$, all the considered usage-rate levels can be selected. Consequently, setups with smaller K are nested within those with larger K.

The cost parameters are: $c_i = 0.1$, $c_p = 7$, $c_c = 17$ and $c_d = 5$. The usage cost rates are: $c_{u,1} = 0$, $c_{u,2} = 1.25$, $c_{u,3} = 1.5$ and $c_{u,4} = 3.75$. For each setup $\mathcal{S}_K$ and each considered time horizon, the optimal maintenance policy $\boldsymbol{\pi}^*$ is obtained through the BRA described in Section 4 and its expected total cost is evaluated via Eq. (21). The discretization of the degradation state is performed with $N + 2 = 102$ representative values.

Table 2. Values of $EC = E\{\mathcal{C}_{T_1}(\boldsymbol{\pi}^*)\}$, f_{PM}, f_{CM}, f_1, f_2, f_3, and f_4 in the case for $T = T_1$.

$\mathcal{S}_K$	EC	f_{PM}	f_{CM}	f_1	f_2	f_3	f_4
$\mathcal{S}_1$	3.51	20.5	4.02	100			
$\mathcal{S}_2$	2.96	1.77	1.03	90.6		9.40	
$\mathcal{S}_3$	2.81	2.78	1.33	89.0	9.1	0.77	1.13

Table 3. Values of $EC = E\{\mathcal{C}_{T_2}(\boldsymbol{\pi}^*)\}$, f_{PM}, f_{CM}, f_1, f_2, f_3, and f_4 in the case for $T = T_2$.

$\mathcal{S}_K$	EC	f_{PM}	f_{CM}	f_1	f_2	f_3	f_4
$\mathcal{S}_1$	6.01	54.6	4.28	100			
$\mathcal{S}_2$	4.49	8.79	2.30	88.3		11.7	
$\mathcal{S}_3$	4.23	1.74	1.26	89.3	0.90	1.0	8.9

Table 4. Values of $EC = E\{\mathcal{C}_{T_3}(\boldsymbol{\pi}^*)\}$, f_{PM}, f_{CM}, f_1, f_2, f_3, and f_4 in the case for $T = T_3$.

$\mathcal{S}_K$	EC	f_{PM}	f_{CM}	f_1	f_2	f_3	f_4
$\mathcal{S}_1$	7.89	87.4	2.14	100			
$\mathcal{S}_2$	7.00	32.6	3.01	86.2		13.8	
$\mathcal{S}_3$	6.34	10.3	2.06	85.0	1.9	2.30	10.8

To evaluate the performance of the policy, we perfomed $N_{\mathrm{S}} = 10^5$ Monte Carlo simulations. Tables 2-4 report the expected total costs of the optimal policies under setups $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ for three different values of the time horizon, T_1, T_2, and T_3,. The same tables also report the values of the indices $f_{\mathrm{PM}} = N_{\mathrm{PM}}/N_{\mathrm{S}} \times 100$ $f_{\mathrm{CM}} = N_{\mathrm{CM}}/N_{\mathrm{S}} \times 100$, and $f_k = N_k/(n \times N_{\mathrm{S}}) \times 100$ ($k = 1, 2, 3, 4$), where N_{PM} is the number of simulations where at least one PM is made, N_{CM} is the number of simulations where at least one CM is made, N_k is the number of times, in N_{S} simulations, at the decision epochs, the usage rate u is set to u_k, and $n \times N_{\mathrm{S}}$ ($n = T/\tau$) is the total number of decisions made in N_{S} simulations. Note that, since τ is always set to 0.5 irrespectively of T, n depends on T.

From the results in Tables 2-4, three main observations emerge:

- For all the considered horizons, setups $\mathcal{S}_2$ and $\mathcal{S}_3$ lead to substantial cost reductions compared with the PdM policy $\mathcal{S}_1$. For T_1, the expected cost decreases from 3.51 to 2.96 and 2.81 (reductions of approximately 15.7% and 19.9%). For T_2, the cost decreases by approximately 25.3% and 29.6%, while for T_3 it decreases by about 11% and 19.6%.
- Most of the benefit is obtained by moving from $\mathcal{S}_1$ to $\mathcal{S}_2$. The additional flexibility in $\mathcal{S}_3$ further reduces the expected cost, but with diminishing returns (approximately 5.25%, 5.79%, and 9.42% reductions relative to $\mathcal{S}_2$ for T_1, T_2, and T_3, respectively). We note that the incremental benefit of adding more usage-rate levels increases with the horizon length in the considered numerical setting.
- In all the considered experimental situations, the system operates predominantly at the nominal level u_1, yet the overall use of the most reduced usage-rate levels increases with the time horizon. Under setup $\mathcal{S}_2$, the reduced level u_3 is selected with frequency $f_3 = 9.40\%$, 11.7%, and 13.8% for T_1, T_2, and T_3, respectively. Under setup $\mathcal{S}_3$, the strongest reduction u_4 is used more ($f_4 = 1.13\%,\ 8.9\%,\ 10.8\%$).

We now focus on the longest time horizon T_3 to better understand why allowing lower-than-nominal usage-rate levels yields such cost reductions.

Figure 1 reports 100 simulated degradation paths up to T_3 in the absence of both replacements and usage-rate reductions. The solid red curve represents the degradation mean given in Eq. (3).

The inverse s-shape of simulated paths

reflects the effects of a bath-tube shaped DR, (Giorgio et al. 2024).

The paths indicated as Case 1, Case 2, and Case 3, in Figure 1, are representative of units whose degradation level is higher than the mean, is lower than the mean, and is close to the mean, respectively. For each of these paths, the quantiles (i.e., the random numbers) used to generate the degradation increments were recorded and subsequently reused to simulate the evolution of the same systems under the optimal policy.

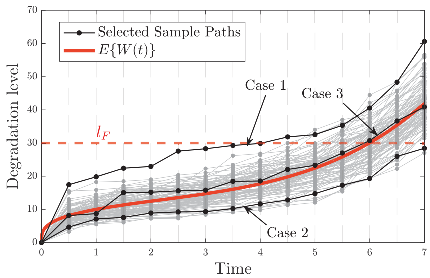

Fig. 1. Simulated degradation paths up to T_3 in the absence of maintenance and prescriptive actions.

Figures 2-4 show what would have been the evolution of the selected paths (in black) under the optimal policy. The scale on the left vertical axis reports the degradation level. Background colors indicate the usage-rate used in each time interval. Namely, green corresponds to u_2, orange to u_3, red to u_4; uncolored intervals correspond to u_1. The dark blue curve represents the DR experienced by the system under the adopted usage-rate, while the light blue curve corresponds to u_1. The scale on the right vertical axis refers to the DR.

At $t_0 = 0$, where the degradation level is assumed to be zero for all the units, the optimal policy always prescribes the lowest usage-rate level u_4. This choice is consistent with the circumstance that under the adopted bathtub-shaped DR, the degradation rate in $(0,\tau)$ is very high, going to infinity as $t \to 0$. In fact, setting the usage-rate to u_4 allows mitigating the risk of accumulating a high degradation level in $(0,\tau)$.

Figure 2 illustrates Case 1 in detail. Here, the first degradation increment is particularly large, and subsequently the degradation level continues to increase rapidly. As the observed degradation becomes sufficiently high, the optimal policy leads to performing a PM.

Figure 3 shows Case 2. In this case, due to the circumstance that the degradation level is always safely far from the failure threshold, after the first interval, the usage rate is always set at the highest level.

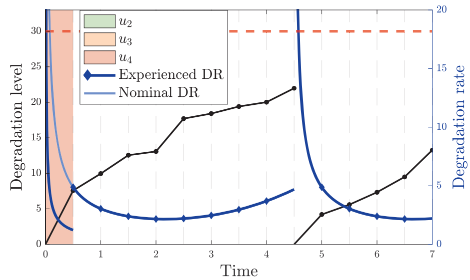

Fig. 2. Degradation path (in black) and DR (in blue) under the optimal policy in Case 1.

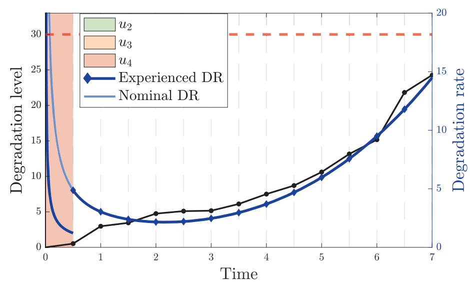

Fig. 3. Degradation path (in black) and DR (in blue) under the optimal policy in Case 2.

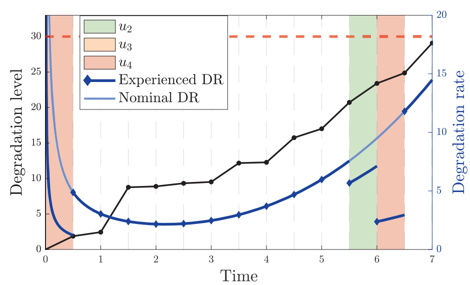

Fig. 4. Degradation path (in black) and DR (in blue) under the optimal policy in Case 3.

Case 3, shown in Figure 4, is the one that best illustrates the benefits of usage-rate flexibility under a bathtub-shaped DR. In fact, in this case, differently from Case 2, as the system approaches the third degradation phase, the policy selectively prescribes lower-than-nominal usage-rate levels when the observed degradation leads to computing a non-negligible probability of reaching the threshold before the end of the mission.

At the last inspection, the policy always prescribes the use of the nominal usage-rate u_1. This circumstance follows from the finite-horizon formulation and the adopted cost structure, which assumes that no maintenance is performed at T_3, even if the unit is failed. Thus, the decision at the last inspection time only aims to find the best trade-off between usage-rate and downtime costs. In fact, the decision to use the nominal usage rate is because the time between inspections is relatively short, so reducing the usage rate would incur a cost that is higher than the expected downtime cost.

6. Conclusion

This work proposes a finite time horizon prescriptive maintenance policy for degrading systems whose degradation rate function exhibits a bathtub-shaped behavior. The proposed policy jointly prescribes maintenance actions and usage-rate reductions, and the optimal solution is obtained by solving the Bellman equation via the backward recursion algorithm. The numerical analysis shows that, under the considered settings, allowing for usage-rate modifications substantially improves the performance respect to a standard predictive maintenance policy, reducing both preventive and corrective maintenance actions across all considered time horizons. In particular, the optimal policy adapts to the degradation rate behavior of the process, favoring in some cases the use of a lower usage-rate level before the first inspection. This approach is consistent with common industrial practice during the run-in phase, where operating conditions are initially moderated to avoid sharp degradation increases at the beginning of the operating life.

To extend the proposed model, in a future work, we intend to consider the possibility of planning the next inspection time together with the maintenance action and the usage-rate level, adaptively, based on inspections outcomes.

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