Proc. of the European Safety and Reliability Conference (ESREL 2026)
Edited by José C. Matos, Paulo B. Lourenço, Michael Beer, Daniel Oliveira,
Edoardo Patelli, Hélder S. Sousa, Rui A. Silva and Monica Santamaria

doi: 10.3850/ESREL2026061419_esrel26-p27880-cd

A prescriptive maintenance policy for a two-unit system in the presence of production constraints

Nicola Esposito

Scuola Superiore Meridionale, Italy. E-mail: n.esposito@ssmeridionale.it

Bruno Castanier

Université d'Angers, France. E-mail: bruno.castanier@univ-angers.fr

Massimiliano Giorgio

Università di Napoli Federico II, Italy. E-mail: massimiliano.giorgio@unina.it

Gianpaolo Pulcini

Istituto di Scienze e Tecnologie per Energia e Mobilità Sostenibili (STEMS), CNR, Napoli, Italia, E-mail: gianpaolo.pulcini@stems.cnr.it

In this paper, we propose a prescriptive maintenance policy for a two-unit system subject to degradation. We suppose that the system is inspected only once during its mission duration, that both mission duration and inspection time are predetermined, and that the inspection allows to know the exact value of the degradation level of the units. It is assumed that units fail when their degradation level exceeds a given failure threshold, that failures are not self-announcing, and that maintenance activities can be performed at the inspection time only. Moreover, it is also assumed that maintenance activities can involve only prescriptive actions that consist in tuning the usage rate of the units when this is deemed convenient. The production rate depends on the usage rate. Higher usage rates incur higher usage costs. On the other hand, reducing the usage rate of a unit slows its degradation process and reduces the risk of failure. The production rate of a failed unit is zero. We assume that there exists a target production rate for the system and that missing the target triggers economic penalties conceived to compensate for the loss of production. Both low usage rates and failures can incur penalties. The tuning of the usage rate is made via a condition-based rule that uses the information gained through the inspection. Objective of the policy is to find the optimal trade-off between usage costs and penalties. The optimal policy is found by minimizing the expected overall cost.

Keywords: Condition-based decision, prescriptive maintenance, usage rate tuning, two-unit system, bivariate gamma degradation process.

1. Introduction

Prescriptive Maintenance (PsM) has recently emerged as a paradigm that enhances the effectiveness of traditional Predictive Maintenance (PdM). While PdM approaches focus on forecasting the future state of a system, PsM (Longhitano et al. 2021) optimizes decision-making by recommending actions that encompass not only standard maintenance interventions - such as inspections and preventive or corrective replacements - but also operational prescriptions about how the system should be operated.

In line with the research route outlined by Longhitano et al. (2021), Esposito et al. (2022), and Esposito et al. (2023a), in this paper we focus on prescriptive actions that consist in modulating the system's usage rate. Specifically, we focus on a two-unit degrading system and assume that the only action that is available at decision epochs consists in adjusting the usage rate of the units, when this is deemed beneficial. Indeed, while a lower usage rate reduces the risk of incurring a failure, it also reduces the production rate. Both usage rate reductions and failures could incur penalties. Moreover, operating the units at higher usage rate incurs usage costs that are not always counterbalanced by added rewards. Consequently, the optimal decision rule must be identified by determining the optimal trade-off between reliability gains, usage cost, rewards, and penalties. The proposed PsM identifies the optimal policy by minimizing the total expected cost in a single mission.

Regarding the penalties, we assume that a contractual clause establishes a target production rate for the system and that missing the target triggers economic penalties conceived to compensate for the loss of production.

In this preliminary study, we focus on a mission-oriented problem and assume that both the mission duration and the inspection time are defined a priori. The inspection is assumed to provide the exact degradation level of both the units. Moreover, we assume that the system can be inspected only once during the mission. In fact, we consider this setting to evaluate the effect of the suggested decision rule, minimizing the risk that it might be masked by the effect of other actions or maintenance decisions.

However, the idea of developing a single-inspection condition-based policy is not new (Crowder and Lawless 2007, Finkelstein et al. 2020, Cha et al. 2022, Esposito et al. 2023b, and Esposito et al. 2026) and can be often justified by using economic arguments (Yuan et al. 2021, Piscopo et al. 2024, and Esposito et al. 2026) especially in sectors, such as offshore wind energy and the petrochemical industry, where very high mobilization costs for specialized technical teams justify the decision of performing all the maintenance tasks at the same inspection epoch.

The degradation process of the units, which are supposed to be identical, is modeled by using two stochastically independent gamma processes. The decision of selecting the gamma process as reference degradation model is due to its widespread adoption and proven flexibility in maintenance modeling (van Noortwijk 2009).

The units are considered failed when their degradation level exceeds a critical threshold.

The rest of the paper is organized as follows. Section 2 describes in detail the proposed policy. Section 3 illustrates the adopted degradation model. Section 4 and 5 describe the formulation of the cost function. Section 6 presents the results of an application example, while Section 7 is devoted to conclusions.

2. The maintenance policy

We consider a system consisting of two identical units that operate over a mission of fixed duration T. At the end of the mission, both units are decommissioned at a fixed cost, regardless of their condition. During the mission, no replacement is allowed. Nevertheless, we suppose that at the predetermined time τ an inspection is performed which returns the exact degradation level of both units and that, based on the output of the inspection, a decision is made about whether to operate the units at their current usage rate or modifying it to mitigate the risk of failure. Reducing the usage rate of a unit slows its degradation process but determines a production loss that might incur penalties if not adequately compensated by the production rate of the other unit. On the other hand, operating a unit at a high usage rate can incur failures that in turn could determine penalties if the other unit works at a usage rate that does not allow the production rate to meet the target. About the mentioned penalties, we assume that a contractual clause establishes a target production rate, say $r_T = 1$, for the system and that missing the target triggers economic penalties, conceived to compensate for the loss of production. The amount of the penalty depends on the difference between the actual production rate of the system and the target.

It is supposed that each unit, at the maximum usage rate, has a production rate that equals the target, so that the system production rate can double the target. Operating both the units at their maximum usage rate could allow the system to satisfy the target even if one of the units fails. However, this option entails usage costs that are not compensated by increased revenue, because exceeding the production target does not result in a bonus or extra reward.

We also suppose that each unit can be either operated at its maximum usage rate (say 1) or at half of it (say 0.5). The usage rates of the units are decided at time zero, where both the units are assumed to be new, and at the inspection time τ, where their optimal values are identified via a condition-based decision rule, which is informed by the output of the inspection.

Units are assumed to fail when their degradation level exceeds an assigned failure threshold. Failures are assumed to be not self-announcing (Bismut et. al. 2022 and Bautista et al. 2022), so that they can be detected only at inspection time or at the end of the mission, when the units are dismounted. Failed units can continue to operate, albeit at production rate 0. Their usage rate can be changed only at the inspection time.

3. The degradation model of the system

Let's start considering the case where the two units in the absence of maintenance actions, are operated at fixed non-null usage rates, say u_1 and u_2. In this case, to describe the evolution of the degradation level of the two units, we use two stochastically independent homogeneous gamma processes, say $\{W_1(t); t \geq 0\}$ and $\{W_2(t); t \geq 0\}$ (Abdel-Hameed 1975).

The homogeneous gamma process is a Markovian stochastic process with stationary and independent increments. Thus, under this setting, given an initial condition, here $W_i(0) = 0\ \forall i \in \{1,2\}$, the process $\{W_i(t); t \geq 0\}$ is fully defined by the distribution of its increment, that is gamma distributed.

By following Tseng et al. (2009), Esposito et al. (2023a), and Esposito et al. (2023b), we express the probability density function (pdf) of the increment $\Delta W_i(t, t+\Delta t) = W_i(t+\Delta t) - W_i(t)$ of the process $\{W_i(t); t \geq 0\}$ as:

$$f_{\Delta W_i(t,t+\Delta t)}(\delta_i) = \frac{\lambda^{a(u_i)\cdot\Delta t} \cdot \delta_i^{a(u_i)\cdot\Delta t-1} \cdot e^{-\lambda\cdot\delta_i}}{\Gamma(a(u_i)\cdot\Delta t)}, \quad \delta_i \geq 0, \quad (1)$$

where $\Gamma(\cdot)$ is the gamma function (viz. $\Gamma(y) = \int_0^\infty z^{y-1} \cdot e^{-z} \mathrm{d}z, y > 0$), and λ and $a(u_i)$ are positive parameters.

Thus, roughly speaking, we assume that $\{W_i(t); t \geq 0\}$ is a gamma process whatever u_i, and that u_i only affects the value of the parameter $a(u_i)$, while λ does not vary with it.

From (1), the marginal cumulative distribution function (cdf) of the increment is:

$$F_{\Delta W_i(t,t+\Delta t)}(\delta_i) = \frac{\gamma(a(u_i)\cdot\Delta t, \lambda\cdot\delta_i)}{\Gamma(a(u_i)\cdot\Delta t)}, \quad (2)$$

where $\gamma(\cdot,\cdot)$, is lower incomplete gamma function (viz., $\gamma(y, b) = \int_0^b z^{y-1} \cdot e^{-z} \mathrm{d}z,\ y > 0$).

From (1) and (2), the pdf and cdf of $W_i(t) = \Delta W_i(0,\ t)$ are:

$$f_{W_i(t)}(w_i) = \frac{\lambda^{a(u_i)\cdot t} \cdot w_i^{a(u_i)\cdot t-1} \cdot e^{-\lambda\cdot w_i}}{\Gamma(a(u_i)\cdot t)}, \quad w_i \geq 0 \quad (3)$$

and

$$F_{W_i(t)}(w_i) = \frac{\gamma(a(u_i)\cdot t, \lambda\cdot w_i)}{\Gamma(a(u_i)\cdot t)}, \quad (4)$$

respectively.

The mean and variance of $W_i(t)$ are:

$$E\{W_i(t)\} = a(u_i)\cdot t/\lambda, \quad (5)$$

$$V\{W_i(t)\} = a(u_i)\cdot t/\lambda^2. \quad (6)$$

From (1) and (3), the conditional cdf of $W_i(t)$, given $W_i(\tau) = w_{i,\tau}$, for $t \leq \tau$, is:

$$F_{W_i(t)|W_i(\tau)}(w_i|w_{i,\tau}) = \mathbb{B}\left(\frac{w_i}{w_{i,\tau}}; a(u_i)\cdot t, a(u_i)\cdot(\tau - t)\right), \quad (7)$$

where $\mathbb{B}(\cdot;\cdot,\ \cdot)$ is the regularized lower incomplete beta function:

$$\mathbb{B}(y; b, c) = \frac{\int_0^y z^{b-1}\cdot z^{c-1}\mathrm{d}z}{B(b,c)}, \quad 0 \leq y \leq 1,$$

with $b, c > 0$, and $B(\cdot,\cdot)$ is the beta function:

$$B(b,c) = \frac{\Gamma(b)\cdot\Gamma(c)}{\Gamma(b+c)}.$$

The joint cdf of $W_1(t_1)$ and $W_2(t_2)$ is:

$$F_{W_1(t_1),W_2(t_2)}(w_1, w_2) = \prod_{i=1}^{2} F_{W_i(t_i)}(w_i) = \prod_{i=1}^{2} \frac{\gamma(a(u_i)\cdot t_i, \lambda\cdot w_i)}{\Gamma(a(u)\cdot t_i)}. \quad (8)$$

Let us denote by $u_{i,j}$ the usage rate of the unit i in the time interval j, where $i = 1$ indicates the unit 1, $i = 2$ indicates the unit 2, $j = 1$ indicates the interval $(0, \tau)$ and $j = 2$ the interval (τ, T). So that, for example $u_{1,2}$ indicates the usage rate adopted for the unit 1 in the time interval (τ, T). To model the effect of maintenance actions we assume that:

***Assumption* 1**

■ Given $u_{1,2}$ and $u_{2,2}$, the evolution of degradation growths of the two units in the time interval (τ, T), say $H_1(\tau, T) = \{\Delta W_1(\tau, t_1);\ \tau < t_1 \leq T\}$ and $H_2(\tau, T) = \{\Delta W_2(\tau, t_2);\ \tau < t_2 \leq T\}$, are conditionally independent of the histories, say $H_1(0, \tau) = \{W_1(t_1); 0 < t_1 \leq \tau\}$ and $H_2(0, \tau) = \{W_2(t_2); 0 < t_2 \leq \tau\}$, of $\{W_1(t_1); t_1 \geq 0\}$ and $\{W_2(t_2); t_1 \geq 0\}$ in $(0, \tau)$.■

Note that, under the policy we propose in this paper, $\Delta W_1(\tau, T)$ and $\Delta W_2(\tau, T)$ are not (marginally) independent of $W_1(\tau)$ and $W_2(\tau)$, because the usage rates $u_{1,2}$ and $u_{2,2}$ are set by using a decision rule that uses as input data $W_1(\tau)$ and $W_2(\tau)$. This in turn also implies that $u_{1,2}$ and $u_{2,2}$ are random variables. Nevertheless, for economy of notation we use the lowercase letters both for them and their realizations. Finally, for simplicity, yet without loss of generality, the link between the parameter $a(u_{i,j})$ and the usage rate $u_{i,j}$ is modeled by using the linear function:

$$a(u_{i,j}) = a_0 \cdot u_{i,j}\,. \qquad (9)$$

In developing the numerical example presented in Section 6, we will assume that both at time $t = 0$ and $t = \tau$, the decision can even be operating only one unit or none of them. In this latter case, to indicate that a unit is not operated we will write that its usage rate is 0. Consequently, we will also assume that if a unit is not operated in $(0, \tau)$ its degradation level at τ is 0 with probability 1.

3.1. Lifetime of the units

As per the classical failure threshold model, we assume that the units fail when their degradation level passes an assigned failure threshold, say D.

Due to the monotonic nature of the gamma process, the cdf of the lifetime of the unit i, say X_i $(i \in (1,2))$, is:

$$F_{X_i}(x) = 1 - F_{W_i(x)}(D).$$

Thus, from (7), the conditional cdf of X_i, given $W_i(\tau) = w_{i,\tau}$, when $w_{i,\tau} > D$, can be expressed as:

$$\begin{aligned} &F_{X_i|W_i(\tau)}(x|w_i) \\ &\quad = 1 - F_{W_i(x)|W_i(\tau)}(D|w_{i,\tau}) \\ &\quad = 1 - \mathbb{B}\left(\frac{D}{w_{i,\tau}}; a(u_{i,1}) \cdot x, a(u_{i,1}) \cdot (\tau - x)\right). \end{aligned} \qquad (10)$$

The (10) can be used for any $x \in (0, \tau)$.

Whereas, from (2), under the Assumption 1, when $w_i \le D$, the conditional cdf of X_i, given $W_i(\tau) = w_{i,\tau}$ and $u_{i,2}$, is:

$$\begin{aligned} &F_{X_i|W_i(\tau)}(x|w_{i,\tau}) \\ &\quad = 1 - F_{\Delta W_i(\tau,x)}(D - w_{i,\tau}) \\ &= 1 - \frac{\gamma\left(a(u_{i,2}) \cdot (x - \tau), \lambda \cdot (D - w_{i,\tau})\right)}{\Gamma\left(a(u_{i,2}) \cdot (x - \tau)\right)}\,. \end{aligned} \qquad (11)$$

The (11) can be used for $\tau < x \le T$.

4. Formulation of the mission function

No replacements are allowed. Costs consist of penalties, which depend on the production rate, and of usage costs, which depend on usage rates. Let $c_{pe}(r(t; \boldsymbol{u}, \boldsymbol{X})) \cdot \mathrm{d}t$ denote the penalty incurred in $(t, t + \mathrm{d}t)$ when the system production rate is $r(t; \boldsymbol{u}, \boldsymbol{X})$, where $\boldsymbol{u} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\boldsymbol{u}_1 = \{u_{1,1}, u_{2,1}\}$, $\boldsymbol{u}_2 = \{u_{1,2}, u_{2,2}\}$, $\boldsymbol{X} = \{X_1, X_2\}$, and X_i is the failure time of the unit i. The production rate of the system is defined as:

$$\begin{aligned} &r(t; \boldsymbol{u}, \boldsymbol{X}) \\ &\quad = \left(r_{1,1} \cdot 1_{\{X_1 > t\}} + r_{2,1} \cdot 1_{\{X_2 > t\}}\right) \cdot 1_{\{t \le \tau\}} \\ &\quad + \left(r_{1,2} \cdot 1_{\{X_1 > t\}} + r_{2,2} \cdot 1_{\{X_2 > t\}}\right) \cdot 1_{\{t > \tau\}}, \end{aligned} \qquad (12)$$

where $r_{i,j}$ is the nominal production rate corresponding to the usage rate $u_{i,j}$. When $u_{i,j}$ is 1, then $r_{i,j}$ is equal to the target production rate r_T, While, when $u_{i,j}$ is 0.5, then $r_{i,j}$ is $r_T/2$. When $u_{i,j}$ is zero, then $r_{i,j}$ is zero. Failed units have production rate 0, irrespective of their usage rate. The expression of $r(t; \boldsymbol{u}, \boldsymbol{X})$ accounts for the effect of failures via the function $1_{\{E\}}$, which takes the value 1 when E is true and 0 otherwise. In this paper, for simplicity, yet without loss of generality, we will set $r_T = 1$. Thus, see (12), $r(t; \boldsymbol{u}, \boldsymbol{X})$ can take values 0, 0.5, 1, 1.5, and 2.

It is assumed that at time $t = 0$ the units are new (i.e., at $t = 0, W_1(0) = W_2(0) = 0$).

The penalty in $(0, \tau)$ is:

$$C_{pe}^{(1)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) = \int_0^\tau c_{pe}(r(t; \boldsymbol{u}, \boldsymbol{X}))\mathrm{d}t, \quad (13)$$

while the penalty in (τ, T) is:

$$C_{pe}^{(2)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) = \int_\tau^T c_{pe}(r(t; \boldsymbol{u}, \boldsymbol{X}))\mathrm{d}t. \quad (14)$$

The usage cost $C_u(\boldsymbol{u}_1, \boldsymbol{u}_2)$, which quantifies the cost of operating the units at rate $\boldsymbol{u}$ over the mission duration, is computed as:

$$\begin{aligned} &C_u(\boldsymbol{u}_1, \boldsymbol{u}_2) \\ &\quad = \sum_{i=1}^{2} c_u(u_{i,1}) \cdot \tau + c_u(u_{i,2}) \cdot (T - \tau). \end{aligned} \quad (15)$$

where $c_u(u)$ is the usage cost rate of a unit that operates at usage rate u.

Notably, the usage cost $C_u(\boldsymbol{u}_1, \boldsymbol{u}_2)$ depends only on $\boldsymbol{u}$ (indicating that usage costs are the same regardless of the failure time of the units). Setting the production rate to zero means that the unit does not operate and incurs a usage cost that is equal to zero.

It is finally assumed that, at the end of the mission, the system is decommissioned at a fixed cost $2 \cdot c_d$, independently of its degradation state. However, if one or both units remain unused throughout the mission (i.e., their usage rate is set equal to 0 over the entire mission), a cost saving equal to c_d is realized for each unused unit.

Hence, the decommission cost $C_d(\boldsymbol{u}_1,\boldsymbol{u}_2)$ is:

$$C_d(\boldsymbol{u}_1,\boldsymbol{u}_2) = c_d \cdot \left(1_{\{u_{1,1}+u_{1,2}>0\}} + 1_{\{u_{2,1}+u_{2,2}>0\}}\right). \quad (16)$$

Thus, the total cost $C_T(\boldsymbol{u}_1,\boldsymbol{u}_2)$ incurred in a mission is:

$$C_T(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{X}) = C_u(\boldsymbol{u}_1,\boldsymbol{u}_2) + C_d(\boldsymbol{u}_1,\boldsymbol{u}_2) + \sum_{j=1}^{2} C_{pe}^{(j)}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{X}) \quad (17)$$

The maintenance policy sets the usage rates $u_{1,1}$ and $u_{2,1}$ to use in $(0,\tau)$ and defines the rates $u_{1,2}$ and $u_{2,2}$ to use in (τ,T) based on the observed states $w_{1,\tau}$ and $w_{2,\tau}$ of the two units at τ. The decision at τ is made by using a two-dimensional rule, say $\mathbb{A}(\boldsymbol{w}_\tau)$, where $\boldsymbol{w}_\tau = \{w_{1,\tau}, w_{2,\tau}\}$, which maps optimal values of $u_{1,2}$ and $u_{2,2}$ as a function of $w_{1,\tau}$ and $w_{2,\tau}$. It is worth pointing out that, due to the Markov assumption, the shape of the function $\mathbb{A}(\boldsymbol{w}_\tau)$, only depends on $\boldsymbol{w}_\tau$, and is independent of the maintenance decisions in the interval $(0,\tau)$.

Objective of the maintenance policy is minimizing the expected mission cost $E\{C_T(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{X})\}$. The optimal policy is found by setting $\boldsymbol{u}_1$ in all the available manners, defining for each of these settings the policy that minimizes the expected mission cost, and selecting among all the resulting policies the one which provides the global minimum.

The function $\mathbb{A}(\boldsymbol{w}_\tau)$ is discretized over a grid partitioning $(0,\infty)^2$, and returns the same action (i.e., values of the usage rates of the two units) for all the points pertaining to the same cell. The optimal policy associates to every cell the action that minimizes the expected mission cost. The proper grid resolution is determined by progressively increasing the density until the mapping exhibits invariance to further refinement, thereby ensuring that the numerical solution is independent of spatial discretization.

5. Formulation of the expected mission cost

The expected mission cost incurred by performing the action $\mathbb{A}_0$ at $t=0$ and using the function $\mathbb{A}(\boldsymbol{w}_\tau)$ at τ is denoted as:

$$E\{C_T(\mathbb{A}_0, \mathbb{A}(\boldsymbol{W}(\tau)), \boldsymbol{X})\}.$$

where $\boldsymbol{W}(\tau) = (W_1(\tau), W_2(\tau))$.

The aim of the policy is defining the action $\mathbb{A}_0^*$ and the function $\mathbb{A}^*(\boldsymbol{w}_\tau)$ that yield the minimum expected mission cost C_T^*.

When $\mathbb{A}_0 = \boldsymbol{u}_1$, the expected mission cost is approximated by using the following formula:

$$E\{C_T(\boldsymbol{u}_1, \mathbb{A}(\boldsymbol{W}(\tau)), \boldsymbol{X})\} = \sum_{m=1}^{M} E\{C_T(\boldsymbol{u}_1, \mathbb{A}(\boldsymbol{W}(\tau)), \boldsymbol{X}) | \boldsymbol{w}_m\} \times \times P(\boldsymbol{W}(\tau) \in Cell_m),$$

where M is the number of cells of the grid, $Cell_m$ is the cell m, and $\boldsymbol{w}_m = (w_{1,m}, w_{2,m})$ is its center. The probability $P(\boldsymbol{W}(\tau) \in Cell_m)$ is computed as:

$$P(\boldsymbol{W}(\tau) \in Cell_m) = \prod_{i=1}^{2} \left[F_{W_i(\tau)}(v_{i,m+1}) - F_{W_i(\tau)}(v_{i,m})\right],$$

where $v_{1,m}$, $v_{1,m+1}$, $v_{2,m}$, $v_{2,m+1}$ are the vertices of the m-th cell, $v_{1,M+1} = v_{2,M+1} = \infty$, and $F_{W_i(\tau)}(\cdot)$ is the cdf (4).

The conditional expected mission cost, given $\boldsymbol{W}(\tau) = \boldsymbol{w}_m$ when $\mathbb{A}_0 = \boldsymbol{u}_1$ and $\mathbb{A}(\boldsymbol{w}_m) = \boldsymbol{u}_2$, is:

$$E\{C_T(\mathbb{A}_0, \mathbb{A}(\boldsymbol{W}(\tau)), \boldsymbol{X}) | \boldsymbol{w}_m\} = C_u(\boldsymbol{u}_1,\boldsymbol{u}_2) + C_d(\boldsymbol{u}_1,\boldsymbol{u}_2) + \sum_{j=1}^{2} E\left\{C_{pe}^{(j)}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{X}) | \boldsymbol{w}_m\right\}.$$

In fact, given $\boldsymbol{W}(\tau) = \boldsymbol{w}_m$, the usage cost $C_u(\boldsymbol{u}_1,\boldsymbol{u}_2)$ and the decommission cost $C_d(\boldsymbol{u}_1,\boldsymbol{u}_2)$ are univocally determined.

Differently, given $\boldsymbol{W}(\tau) = \boldsymbol{w}_m$, when $\mathbb{A}_0 = \boldsymbol{u}_1$ and $\mathbb{A}(\boldsymbol{w}_m) = \boldsymbol{u}_2$ $C_{pe}^{(1)}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{X})$ and $C_{pe}^{(2)}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{X})$ still depend on $\boldsymbol{X}$. Thus, given that the pdf of $\boldsymbol{X}$ depends on $\boldsymbol{w}_m$, from (13) and (14) their expected values can be computed as:

$$E\left\{C_{pe}^{(1)}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{X}) | \boldsymbol{w}_m\right\} = E\left\{\int_0^\tau c_{pe}\big(r(t;\boldsymbol{u},\boldsymbol{X})\big) \cdot \mathrm{d}t \,\middle|\, \boldsymbol{w}_m\right\} \quad (18)$$

and

$$E\left\{C_{pe}^{(2)}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{X}) | \boldsymbol{w}_m\right\} = E\left\{\int_\tau^T c_{pe}\big(r(t;\boldsymbol{u},\boldsymbol{X})\big) \cdot \mathrm{d}t \,\middle|\, \boldsymbol{w}_m\right\}. \quad (19)$$

In both these equations $r(t;\boldsymbol{u},\boldsymbol{X})$ is determined as per Eq. (12).

The conditional expected costs (18) and (19) can be computed by using the following expressions:

■ $w_{1,m} \leq D$ and $w_{2,m} \leq D$

$$E\left\{C_{pe}^{(1)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) \middle| \boldsymbol{w}_m\right\} = c_{pe}\left(r_{1,1} + r_{2,1}\right) \cdot \tau$$

and

$$E\left\{C_{pe}^{(2)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) \middle| \boldsymbol{w}_m\right\} = c_{pe}\left(r_{1,2} + r_{2,2}\right) \cdot (T - \tau)$$

$$+ \left(c_{pe}(0) - c_{pe}\left(r_{1,2} + r_{2,2}\right)\right) \cdot \int_{\tau}^{T} \prod_{i=1}^{2} F_{X_i|W_i(\tau)}\left(x \middle| w_{i,m}\right) \mathrm{d}x$$

$$+ \left(c_{pe}\left(r_{2,2}\right) - c_{pe}\left(r_{1,2} + r_{2,2}\right)\right) \cdot \int_{\tau}^{T} \left[F_{X_1|W_1(\tau)}\left(x \middle| w_{1,m}\right) - \prod_{i=1}^{2} F_{X_i|W_i(\tau)}\left(x \middle| w_{i,m}\right)\right] \mathrm{d}x$$

$$+ \left(c_{pe}\left(r_{1,2}\right) - c_{pe}\left(r_{1,2} + r_{2,2}\right)\right) \cdot \int_{\tau}^{T} \left[F_{X_2|W_2(\tau)}\left(x \middle| w_{2,m}\right) - \prod_{i=1}^{2} F_{X_i|W_i(\tau)}\left(x \middle| w_{i,m}\right)\right] \mathrm{d}x$$

■ $w_{1,m} > D$ and $w_{2,m} \leq D$

$$E\left\{C_{pe}^{(1)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) \middle| \boldsymbol{w}_m\right\}$$

$$= c_{pe}\left(r_{1,1} + r_{2,1}\right) \cdot \tau + \left(c_{pe}\left(r_{2,1}\right) - c_{pe}\left(r_{1,1} + r_{2,1}\right)\right) \cdot \int_{0}^{\tau} F_{X_1|W_1(\tau)}\left(x \middle| w_{1,m}\right) \mathrm{d}x$$

and

$$E\left\{C_{pe}^{(2)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) \middle| \boldsymbol{w}_m\right\} = c_{pe}\left(r_{2,2}\right) \cdot (T - \tau) + \left(c_{pe}(0) - c_{pe}\left(r_{2,2}\right)\right) \cdot \int_{\tau}^{T} F_{X_2|W_2(\tau)}\left(x \middle| w_{2,m}\right) \mathrm{d}x$$

■ $w_{1,m} \leq D$ and $w_{2,m} > D$

$$E\left\{C_{pe}^{(1)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) \middle| \boldsymbol{w}_m\right\}$$

$$= c_{pe}\left(r_{1,1} + r_{2,1}\right) \cdot \tau + \left(c_{pe}\left(r_{1,1}\right) - c_{pe}\left(r_{1,1} + r_{2,1}\right)\right) \cdot \int_{0}^{\tau} F_{X_2|W_2(\tau)}\left(x \middle| w_{2,m}\right) \mathrm{d}x$$

and

$$E\left\{C_{pe}^{(2)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) \middle| \boldsymbol{w}_m\right\} = c_{pe}\left(r_{1,2}\right) \cdot (T - \tau) + \left(c_{pe}(0) - c_{pe}\left(r_{1,2}\right)\right) \cdot \int_{\tau}^{T} F_{X_1|W_1(\tau)}\left(x \middle| w_{1,m}\right) \mathrm{d}x$$

■ $w_{1,m} > D$ and $w_{2,m} > D$

$$E\left\{C_{pe}^{(1)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) \middle| \boldsymbol{w}_m\right\}$$

$$= c_{pe}\left(r_{1,2} + r_{2,2}\right) \cdot \tau + \left(c_{pe}(0) - c_{pe}\left(r_{1,2} + r_{2,2}\right)\right) \cdot \int_{0}^{\tau} \prod_{i=1}^{2} F_{X_i|W_i(\tau)}\left(x \middle| w_{i,m}\right) \mathrm{d}x$$

$$+ \left(c_{pe}\left(r_{2,2}\right) - c_{pe}\left(r_{1,2} + r_{2,2}\right)\right) \cdot \left[\int_{0}^{\tau} F_{X_1|W_1(\tau)}\left(x \middle| w_{1,m}\right) - \left(\prod_{i=1}^{2} F_{X_i|W_i(\tau)}\left(x \middle| w_{i,m}\right)\right) \mathrm{d}x\right]$$

$$+ \left(c_{pe}\left(r_{1,2}\right) - c_{pe}\left(r_{1,2} + r_{2,2}\right)\right) \cdot \left[\int_{0}^{\tau} F_{X_2|W_2(\tau)}\left(x \middle| w_{2,m}\right) - \left(\prod_{i=1}^{2} F_{X_i|W_i(\tau)}\left(x \middle| w_{i,m}\right)\right) \mathrm{d}x\right]$$

and

$$E\left\{C_{pe}^{(2)}(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{X}) \middle| \boldsymbol{w}_m\right\} = c_{pe}(0) \cdot (T - \tau)$$

Details about the derivations of these formulas are not reported in this paper due to space constraints.

6. Numerical example

In this section, we present an example of application of the proposed maintenance policy.

Table 1 reports the values we used for the parameters of the cost and degradation models. The mission length is set to $T = 100$ and the inspection time is $\tau = T/2 = 50$.

Table 2 lists all the actions available at the decision epochs, $t = 0$ and $t = \tau$. Thus, for example performing the action $\mathcal{A}_6$ at $t = 0$ consists in setting $u_{1,1} = 0.5$ and $u_{2,1} = 0$.

Table 1. Parameters used in the numerical example.

a_0	λ	D	c_d	$c_u(0)$	$c_u(0.5)$
0.17	0.425	35	50	0	0.02
$c_u(1)$	$c_{pe}(0)$	$c_{pe}(0.5)$	$c_{pe}(1)$	$c_{pe}(1.5)$	$c_{pe}(2)$
0.05	25	15	0	0	0

Table 2. Actions available at decision epochs.

Usage rate					
Action	Unit		Action	Unit	
	1	2		1	2
$\mathcal{A}_1$	1	1	$\mathcal{A}_6$	0.5	0
$\mathcal{A}_2$	1	0.5	$\mathcal{A}_7$	0	1
$\mathcal{A}_3$	1	0	$\mathcal{A}_8$	0	0.5
$\mathcal{A}_4$	0.5	1	$\mathcal{A}_9$	0	0
$\mathcal{A}_5$	0.5	0.5			

We have applied the proposed policy under this setting. It resulted that the optimal action $\mathbb{A}_0^*$ to perform at $t = 0$ is either $\mathcal{A}_3$ or the symmetrical one $\mathcal{A}_7$, and that the optimal decision function $\mathbb{A}^*(\boldsymbol{w}_\tau)$ to use at τ is the one represented in Figure 1, where $w_{1,\tau}$ is on the x-axis, $w_{2,\tau}$ is on the y-axis, and the color of each area in the plane denotes the optimal action $\mathbb{A}^*(\boldsymbol{w}_\tau)$ to implement based on $(w_{1,\tau}, w_{2,\tau})$. The symmetry between $\mathcal{A}_3$ and $\mathcal{A}_7$, as well as the symmetry of the decision function $\mathbb{A}^*(\boldsymbol{w}_\tau)$ with respect to the line $w_{1,\tau} = w_{2,\tau}$ was expected, because the units are identical.

The expected mission cost incurred by using the optimal policy is $C_T^* = 108.5$.

When $w_{1,\tau}$ and $w_{2,\tau}$ are low and comparable, $\mathbb{A}^*(\boldsymbol{w}_\tau) = \mathcal{A}_1$. As $w_{1,\tau}$ and $w_{2,\tau}$ increase, but remain close to each other, the optimal action becomes $\mathcal{A}_5$. Both $\mathcal{A}_1$ and $\mathcal{A}_5$ consist in operating the units at the same usage rate. In contrast, when there is a significant difference between $w_{1,\tau}$ and $w_{2,\tau}$, the optimal action becomes $\mathcal{A}_4$ if $w_{1,\tau}$ is significantly greater than $w_{2,\tau}$, while in the symmetric case the optimal action is $\mathcal{A}_2$. The actions $\mathcal{A}_7$ and $\mathcal{A}_3$ are applied only when at least one between $w_{1,\tau}$ and $w_{2,\tau}$ approaches or exceeds the failure threshold. The actions $\mathcal{A}_6$ and $\mathcal{A}_8$ are never selected. Finally, action $\mathcal{A}_9$ is adopted exclusively when $w_{1,\tau}$ and $w_{2,\tau}$ both exceed the failure threshold (to save on usage costs).

Adopting at $t = 0$ the action $\mathcal{A}_3$ or $\mathcal{A}_7$ implies that in $(0, \tau)$ only one unit is operated.

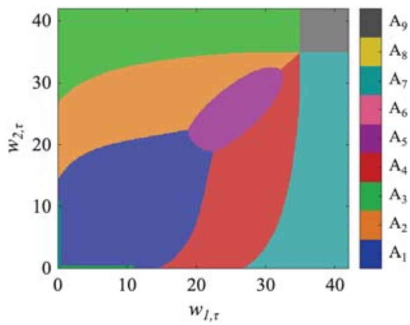

Fig. 1. Shape of $\mathbb{A}^*(\boldsymbol{w}_\tau)$.

To allow a deeper insight into the features of the function $\mathbb{A}^*(\boldsymbol{w}_\tau)$, let us focus on the optimal policy obtained by setting $\mathbb{A}_0^* = \mathcal{A}_3$. In this case, given that the unit 2 is not operated, at τ it will certainly be $w_{2,\tau} = 0$. Consequently, the decision to be taken at $t = \tau$ depends only on $w_{1,\tau}$.

Figure 2 shows the behavior of the function $\mathbb{A}^*(\boldsymbol{w}_\tau)$, obtained when $\boldsymbol{w}_\tau = \{w_{1,\tau}, 0\}$, together with the pdf $f_{W_1(\tau)}(w_{1,\tau})$ (see Eq. (3)) of the degradation state of the unit 1 at time τ. The colored area under the curve are the probabilities that $\mathbb{A}^*(\boldsymbol{w}_\tau)$ is the action corresponding to considered color.

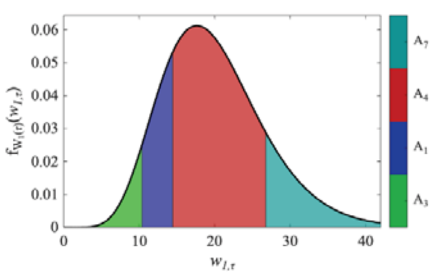

Fig. 2. Optimal action to undertake at τ as a function of $w_{1,\tau}$ and pdf $f_{W_1(\tau)}(w_{1,\tau})$

Figure 2 shows that, when $w_{1,\tau}$ is low, the optimal action at τ is still $\mathcal{A}_3$, which consists in continuing to operate the unit 1 at the maximum usage rate. In this case the unit 2 remains inactive. This decision is motivated by the objective of saving the cost c_d. As $w_{1,\tau}$ increases, the policy selects actions that progressively involve the unit 2. Initially, the policy employs both units at their maximum usage rate (action $\mathcal{A}_1$). For higher values of $w_{1,\tau}$, the usage rate of the unit 1 is gradually reduced, with the optimal action shifting first to $\mathcal{A}_4$ and eventually to $\mathcal{A}_7$.

To evaluate the effectiveness of the proposed policy we have computed the expected mission cost of policies that set all the usage rates a priori. The best of these latter policies has turned out to be the one that consists in operating, at the maximum usage rate, the unit 1 up to τ and the unit 2 from τ to T. Under this policy the expected mission cost is $\tilde{C}_T = 110 - c_i$, where c_i is the inspection cost, which is not incurred by this simple policy. Thus, since the expected mission cost of the proposed prescriptive policy is $C_T^* = 108.5$, then the cost saving $\tilde{C}_T - C_T^* = 1.5 - c_i$ is positive unless $c_i \geq 1.5$. We also report that the expected mission cost that yields from using only one unit at its maximum usage rate all over the mission duration is $\ddot{C}_T = 419$.

7. Conclusions

This paper proposed a prescriptive maintenance policy for a two-unit degrading system working under production constraints, in which maintenance decisions are limited to adjusting the usage rates of the units at a single, predetermined inspection time. By explicitly accounting for the influence of the usage rate on usage costs, failure risk, and penalties arising from unmet production targets, the policy finds optimal condition-based prescriptions through the minimization of the expected total mission cost. The numerical example highlights the properties of the optimal policy and illustrates how inspection information enables reducing the expected mission cost, even by using a single inspection only, with respect to policies that set the usage rates a priori.

References

Longhitano, D., K. Tidriri, C. Bérenguer, and B. Echard (2021). Proposition of a generic decision framework for prescriptive maintenance. In *World congress on engineering asset management*, pp. 263-273. Springer International Publishing.

Esposito, N., B. Castanier, and M. Giorgio (2022). Prescriptive block replacement policy for production degrading systems. *IFAC-PapersOnLine 55(10)*, 1974-1979.

Esposito, N., B. Castanier, and M. Giorgio (2023a). An adaptive prescriptive maintenance policy for a gamma deteriorating unit. In *Proceedings of the33rd European Safety and Reliability Conference (ESREL2023)*, pp. 960-967. Research Publishing, Singapore.

Crowder, M. and J. Lawless (2007). On a scheme for predictive maintenance. *European Journal of Operational Research 176(3)*, 1713-1722.

Finkelstein, M., J. H. Cha, and G. Levitin. (2020). On a new age-replacement policy for items with observed stochastic degradation. *Quality and Reliability Engineering International, 36(3)*, 1132-1143.

Cha, J. H., M. Finkelstein, and G. Levitin (2022). Replacement policy for heterogeneous items subject to gamma degradation processes. *Methodology and Computing in Applied Probability 24*, 1323–1340.

Esposito, N., A. Mele, B. Castanier, and M. Giorgio (2023b). A hybrid maintenance policy for a deteriorating unit in the presence of three forms of variability. *Reliability Engineering & System Safety 237, 109320*, 1–15.

Esposito, N., A. Piscopo, B. Castanier, and M. Giorgio (2026). An adaptive maintenance policy for degrading units with an imperfect and a perfect inspection. *European Journal of Operational Research 330(1)*, 139-156.

Yuan, X.X., E. Higo, and M. Pandey (2021). Estimation of the value of an inspection and maintenance program: a Bayesian gamma process model. *Reliability Engineering & System Safety 216, 107812*, 1-14.

Piscopo, A., N. Esposito, B. Castanier, and M. Giorgio (2024). Maintenance of degrading units: new policy and some critical considerations about the value of inspections. In proceedings of the 34th (ESREL2024), 145-154. Polish Safety and Reliability Association.

Van Noortwijk, J. M. (2009). A survey of the application of gamma process in maintenance. *Reliability Engineering & System Safety 94(1)*, 2-21.

Bismut, E., M. D. Pandey, and D. Straub (2022). Reliability based inspection and maintenance planning of a nuclear feeder piping system. *Reliability Engineering & System Safety, 224, 108521*, 1-16.

Bautista, L., I.T. Castro, and L. Landesa (2022). Condition-based maintenance for a system subject to multiple degradation processes with stochastic arrival intensity. *European Journal of Operational Research 302(2),* 560-574.

Abdel-Hameed, M. (1975). A Gamma Wear Process. IEEE *Transactions on Reliability R-24(2)*, 152-153.,

Tseng, S. T., N. Balakrishnan, and C. C. Tsai (2009). Optimal step-stress accelerated degradation test plan for gamma degradation processes. *IEEE Transactions on Reliability 58(4)*, 611-618.